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Mathematical Tables
and other
Aids to Computation

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THE ELECTROMATIC TABLE PRINTING MACHINE

The Printing of Mathematical Tables

1. Introduction. The printing of extensive mathematical tables requires perfection in accuracy and legibility, and economy of cost and skilled personnel. The printing of nineteen successive volumes of the *American Air Almanac* since 1940 has brought forward several notable developments in the art as well as considerable interesting data on the comparison of different methods. The most recent development is a new table printing machine which combines the attributes of economy and perfection.

The printing of the *American Air Almanac* is a task of some magnitude, considering the high standards of perfection which are required. There are three volumes each year with a total of 730 pages each containing about 3,000 figures. Some editions have run to nearly two hundred thousand copies and it has been necessary on occasion to make successive reprints of a given volume. The assurance of perfection in millions of copies with billions of figures requires great care, especially under wartime printing conditions.

The need for perfection in the *American Air Almanac* is obvious. An erroneous figure could cause the loss of a valuable plane and crew. The figures must be easily legible since the aviator must read them quickly in a bouncing plane with poor illumination when he is tired, cold and short of oxygen. The volumes must be compact since they must be used in cramped quarters.

While the cost in money of the Air Almanac is not the primary consideration, the possible economies are not negligible, and the shortage of competent personnel during the war made the economy of effort a prime factor in providing adequate tables at the proper time. The time factor is important since the possibility of change in the Almanac makes it undesirable to prepare the volumes too far in advance.

2. Methods Used. In printing the nineteen successive volumes of the Air Almanac, four different methods have been used and complete records kept so that objective comparison is possible. These methods include both the setting of type and the preparation of manuscript for photographic reproduction without the use of type. The photographic reproduction has been done by the lithographic process and also by the use of zinc etchings or line cuts. The manuscript has been prepared both by a special process using an electric punched card accounting machine and by the new device recently built for the Naval Observatory by the International Business Machines Corporation.

The three volumes for 1941 were printed from movable type. As is well known the art of printing from such type is very highly developed and the impressions obtained are of the highest perfection in sharpness and clarity. The loose type and rules permit many refinements of composition, but the method is not economical either in cost or in the efforts required of the authors to secure perfection. Not only are errors frequent in setting the type but there is the possibility of error both while corrections are being made and even during the printing itself. In reading the first proofs of these 730 pages 840 errors were found. These were marked for correction,

but when the revised proofs were examined, 68 errors were still present, many of them due to faulty correction of the previous errors.

The 840 errors in the first proofs averaged only a little over one to a page, which is quite acceptable when tables are computed and the copy prepared by hand, since the copy itself frequently contains about the same number of errors. The checks by differencing applied to such tables locate both types of error at the same time. In the case of the Air Almanac, however, the computation and preparation of the copy was performed automatically by means of punched card machines, which gave results of a much higher order of accuracy. The checks by differencing in this case revealed errors almost entirely due to the printer, only about one error in a thousand being due to the machines. The successive proof-reading of all the figures, to eliminate typographical errors from such perfect data, is not only costly but is demoralizing.

For several reasons the books for 1941 were printed directly from the movable type, although it is customary, with large editions or where perfection requires it, to make electroplates from the type and to print from these plates. The use of such plates insures great reliability provided the plate proofs are adequately checked. Electroplates are used on the *American Nautical Almanac* and on the *American Ephemeris and Nautical Almanac.*

In addition to the method described above of setting the figures in type, many tables are printed by processes involving photography from manuscript prepared in other ways. The Yale zone catalogs, for example, have been prepared in this manner from typewritten manuscript. The zone published in 1925 was prepared with an ordinary typewriter, and the later zones with a special machine used only for this purpose. A number of typewriters have been produced in recent years with this application in mind. These machines give remarkably clear images, and one of them automatically regulates the force applied to each character.

The use of typewritten manuscript instead of loose type eliminates the necessity of repeatedly checking all the figures, since any change in the original manuscript is easily detected. The skill of the typist of course determines the accuracy of the original typing, and the problem should be roughly comparable to that of a compositor operating a monotype keyboard machine.

Since the adaptation of punched card accounting machines to scientific work there has been a great deal of copy for reproduction prepared on the printing units of these machines.[1] The speed and accuracy of these printing units is phenomenal, but the printed results generally lack elegance. The individual characters are not as sharp and clear as printed characters and they are so spread out that the tables tend to be bulky. In 1941 a technique was developed in the Nautical Almanac Office which made it possible to overcome these difficulties. This method was used on the twelve volumes of the Air Almanac from 1942 to 1945, and while the expert quickly notices the lack of perfection, the casual reader does not notice that the tables were not printed from type. The degree of economy and perfection attained in these tables is remarkable and has attracted widespread attention.

The outstanding feature of the method is that of printing alternate columns of the table on two successive runs of the machine, thus permitting the printing of twelve characters to the inch in the line instead of six.[2]

The line spacing was also changed for compactness. Special forms were ruled for the process by the Standard Register Company. The clarity of the individual characters was improved by careful examination of several details of the printing. The greatest improvement came from the use of carbon sheets instead of a carbon ribbon; this eliminated broken figures and shadows due to whipping of the ribbon. Careful selection of the grade of carbon paper and the softness of the platen greatly improved the clarity. A special device which applied additional mass behind the platen eliminated the multiple images due to bouncing. Painstaking adjustment of the typebars and spring tension by the Customer Engineers gave optimum registration.

While the method was considered only as a makeshift, it permitted production without special engineering and delay during a very critical period. The twelve volumes produced by this method have well served their purpose.

3. The New Machine. At the time the above method was adopted, another better method was proposed which has now materialized, and the machine is in operation. In this machine a special typewriter designed for printing tables is operated from punched cards. It combines the elegance of the finest typewriter with the accuracy and efficiency of the punched cards. This machine was built by the International Business Machines Corporation and installed in February, 1945. It was used in the production of the volumes of the Air Almanac for 1946, and the results are most gratifying.

The new machine was designed with maximum flexibility so as to print almost any kind of mathematical table. It is so arranged that most of the details of printing can be decided upon after the cards have been prepared, the principal exception being that the data must be printed in the order in which they appear on the card. The style and arrangement of the printing is determined by means of a flexible plugboard and a specially punched master card. Ample provision is made for both vertical and horizontal spacing, for the selection of type styles and symbols, for the suppression of insignificant zeros, and for the arbitrary suppression of any characters on the cards. Such operations as rounding the computed figures to the nearest unit, quarter unit, or other arbitrary division is accomplished with a small amount of gang punching.

The escapement of the typewriter permits spacing in the line in units of $\frac{1}{32}$ of an inch, and the standard ratchet on the platen gives line spacing in units of $\frac{1}{12}$ inch. The usual reduction in the photographic process will be to $\frac{2}{3}$. Three units to a figure thus gives tables which very closely resemble 8-point type, set 16 figures to the inch. The use of a platen ratchet with a different number of teeth would permit the simulation of "leaded" tables. The small figures printed with two units are equivalent to 6-point type, set 24 figures to the inch. The use of a different reduction ratio in the photographic process gives type results resembling those from different size type.

There are 88 characters which, after photographic reduction, give 8-point light figures, 8-point bold figures, 8- and 6-point figures on the half line (without moving platen), 6-point figures on the line, various kinds of decimals, plus and minus signs, etc., Roman numerals, and special symbols for the Air Almanac. Figure 1 shows the characters on the various typebars.

The selection of the basic style of the characters and the modifications of them were made with due consideration of the operation of the typewriter

NS0123456789· 0123456789. NS0123456789. _*■‡°′″()

IVXLUmdhms +-0123456789 +-0123456789 _////□†·−+*

FIGURE 1. Characters on typewriter.

488 GREENWICH P. M. 1941

GCT	☉ SUN GHA	Dec.	♈ GHA	♀ VENUS −3.4 GHA	Dec.	♂ MARS −1.8 GHA	Dec.
h m	° ′	° ′	° ′	° ′	° ′	° ′	° ′
12 00	0 00	N8 20	160 15	327 29	S5 08	136 49	N4 57
10	2 30		162 46	329 59		139 20	
20	5 00		165 16	332 29		141 50	
30	7 30	· ·	167 47	334 59	· ·	144 21	· ·
40	10 00		170 17	337 29		146 51	
50	12 30		172 48	339 59		149 21	
13 00	15 00	N8 19	175 18	342 28	S5 09	151 52	N4 57
10	17 30		177 48	344 58		154 22	
20	20 00		180 19	347 28		156 53	
30	22 30	· ·	182 49	349 58	· ·	159 23	· ·
40	25 00		185 20	352 28		161 53	
50	27 30		187 50	354 58		164 24	

86 GREENWICH P. M. 1944

GCT	☉ SUN GHA	Dec.	♈ GHA	VENUS −3.5 GHA	Dec.	MARS 0.2 GHA	Dec.
h m	° ′	° ′	° ′	° ′	° ′	° ′	° ′
12 00	356 25	S13 58	321 25	31 13	S21 35	252 34	N24 33
10	358 55		323 56	33 43		255 04	
20	1 25		326 26	36 13		257 34	
30	3 55	· ·	328 57	38 42	· ·	260 04	· ·
40	6 25		331 27	41 12		262 35	
50	8 55		333 57	43 42		265 05	
13 00	11 25	S13 57	336 28	46 12	S21 35	267 35	N24 33
10	13 55		338 58	48 42		270 05	
20	16 25		341 29	51 12		272 36	
30	18 55	· ·	343 59	53 42	· ·	275 06	· ·
40	21 25		346 30	56 12		277 36	
50	23 55		349 00	58 41		280 07	

48 GREENWICH P. M. 1946

GCT	☉ SUN GHA	Dec.	♈ GHA	MARS −1.0 GHA	Dec.	JUPITER −1.6 GHA	Dec.
h m	° ′	° ′	° ′	° ′	° ′	° ′	° ′
12 00	356 58	S19 17	303 12	191 40	N26 17	97 47	S 9 06
10	359 28		305 43	194 10		100 17	
20	1 58		308 13	196 41		102 48	
30	4 28	· ·	310 43	199 11	· ·	105 18	· ·
40	6 58		313 14	201 42		107 48	
50	9 28		315 44	204 12		110 19	
13 00	11 58	S19 16	318 15	206 43	N26 17	112 49	S 9 06
10	14 28		320 45	209 14		115 20	
20	16 58		323 15	211 44		117 50	
30	19 28	· ·	325 46	214 15	· ·	120 20	· ·
40	21 58		328 16	216 45		122 51	
50	24 28		330 47	219 16		125 21	

FIGURE 2. Reproductions from the *American Air Almanac* printed from: (upper) movable type, (middle) accounting machine record, (lower) electromatic machine.

and the reproduction by photography. The primary consideration was that of legibility, though beauty was not ignored. For example, it is considered important to be able to recognize a numeral when a portion of it has been lost or when a black speck has been added to it in the printing process. The numerals and most of the letters are based on the characters known as Bell Gothic which were developed by the Mergenthaler Linotype Company for use in telephone directories.

The typewriter is an Electromatic Proportional Spacing Machine. This machine not only permits arbitrary spacing between the columns, but the printing is of unusual elegance. The Electromatic feature insures uniformity in the strength of the impressions. The arbitrary spacing between columns enables one to group figures into degrees, minutes, and seconds and also to space over single or double rules of various widths; added to this is the ability to print 8- and 6-point figures in 3- and 2-unit spaces, respectively, which permits printing with the compactness required in a wide variety of tables.

Figure 2 shows small specimens of the work produced by each of the three methods described above.

4. Printing Methods. Modern printing methods are highly developed and very reliable, but the process is not as nearly perfect as many suppose; eternal vigilance is the price of perfection in printing as well as in computing and proofreading. The fact that the final proofs or the manuscript for photographic reproduction are perfect does not insure perfection in the edition. It is necessary to examine press sheets selected according to the method of printing employed.

Printing by letterpress with electroplates is extremely reliable. One set of plates, barring accident, will print a large edition of several hundred thousand sheets. Battered plates are comparatively rare, so it is feasible to keep track of all such cases, and three press sheets, from the beginning, middle, and end of the run afford a reliable indication of the quality of the edition.

The lithographic or offset process is fast and economical, and under the best conditions gives results of remarkably fine quality. It is nevertheless a more delicate process and generally less reliable than the above method. The offset plates are cheap and easy to make but also easy to spoil; the result is a tendency to use numerous plates on a given job with consequent difficulty of control. The albumen plates are very satisfactory for small editions but require so many different plates for an edition the size of a recent issue of the Air Almanac that it is almost impossible to keep track of them. Even with "deep etched" plates, the chances of printing an entire edition from one set of plates are not good. The lithographic process is so sensitive to quality of materials and workmanship that during 1944 it was not considered feasible to print the Air Almanac by offset with the materials permitted by the War Production Board. Accordingly the last two volumes for 1945 were printed from electroplates made from line cuts. Another advantage of the electroplates is that the final proof is taken from the plate itself whereas in lithography the proof is taken from the negative and the plate or plates are prepared after the proofs are submitted.

A very common fallacy is the belief that lithographic plates and line cuts are exactly like the original, since they are made by a photographic

process. This is erroneous for two reasons. In the first place different degrees of blackness on the manuscript do not show up on the negative in the same proportion. Thus a portion of a figure which looked gray on the manuscript may be entirely missing on the negative. In the second place considerable hand work is done on the negatives. In the lithographic process the retoucher paints over parts of the negative and clears out others. In the case of the line cut the photoengraver "routs" out part of the plate with an implement like a dentist's drill. It is true that with these processes, omissions are more probable than changes, but even these are not impossible.

5. Mechanical Proofreading. The application of the punched card technique has completely revolutionized the methods of proofreading in the Nautical Almanac Office both for tables set in type and for tables printed by methods involving photography; the new methods give revolutionary dependability and economy. The conventional method of having a proofreader assume responsibility for the agreement of the copy and the proof has been abandoned. The proofs, but not the copy, are now given to a card punch operator whose only responsibility is to punch what she sees. A template is used which exposes only one line at a time, so there is no temptation to correct an error by looking up or down the column. These cards are then compared electrically with the cards from which the copy was prepared. This comparison is performed on one of the standard punched card machines at the rate of 6,000 cards per hour. Finally the various columns are differenced automatically on the accounting machine and the differences printed in such a manner that any error would be immediately apparent.

In the case of manuscript prepared on the accounting machine or on the new table printer, the results are so accurate that the line-cut proofs are examined only for general quality and legibility, no attention being paid to the figures themselves. The plate proofs only are subject to the rigorous methods above described.

6. Acknowledgments. The success of various phases of the work has been made possible by the cooperation of many people, including the following: the staff of the Government Printing Office, particularly Mr. R. W. Christie; the staff of the Nautical Almanac Office, especially Mr. G. M. Clemence and Mr. Jack Belzer; Mr. E. W. Gardinor and other engineers of the International Business Machines Corporation; Mr. C. H. Griffith, Mergenthaler Linotype Company.

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[1] See W. J. ECKERT, *Punched Card Methods in Scientific Computation*, The Thomas J. Watson Astronomical Computing Bureau, 1940, p. 106.
[2] See W. J. ECKERT, "Air Almanacs," *Sky and Telescope*, v. 4, Nov. 1944, p. 12.

RECENT MATHEMATICAL TABLES

337[A, B, C, D, N, O].—CLEMENTE BONFIGLI **a**. *Manuale Logaritmico Completo del Tecnico ad uso degli Ingegneri, Architetti, Geometri, Periti, ecc. con cinque cifre decimali Tavole logaritmiche dei numeri; Tavole log. delle Funzioni di angoli sessagesimali, centesimali e sessadecimali; Tavole*

aritmetiche e di trasformazione angolare; Tavole dei val. naturali delle funzioni di angoli sessagesimali e centesimali; Tavole speciali per il calcolo diretto dei prodotti a sen b e a cos b; Tavole dei coefficienti finanziari, a 8 decimali, e loro logaritmi; Tavole per il picchettamento delle curve circolari. Milan, Hoepli, 1945. xvi, 374 p. 12.8 × 19.6 cm. 250 lire, bound in boards.

b. *Nuove Tavole Logaritmiche con cinque decimali (per i sistemi: sessagesimale, centesimale, sessadecimale), Tavole Aritmetiche, Tavole Finanziarie, Valori Naturali, ad uso dei Licei Scientifici e Classici.* Milan, Hoepli, 1944. xvi, 230 p. 12.8 × 19.4 cm. 120 lire, paper covers.

a: Tables, p. 1–337; explanations for the use of the tables p. 340–373. **b**: Tables, p. 1–197; explanations, p. 199–229. Several of the tables in this volume are the same as those in **a**.

338[B].—L. Îa. Neĭshuler, "Ob optimal'nykh slitnykh Tablitsakh kvadratov i kubov" (On optimal coalescent tables of squares and cubes), Akad. N., SSSR, Moscow, (*Dok.*) *C.R.*, n.s., v. 47, 1945, p. 478–481 + folding plate. Russian ed. v. 47, p. 462–465. 16.8 × 25.9 cm. See *MTAC*, v. 1, p. 7, **67**, **68**.

Without questioning the author's right to coin a word or a phrase when introducing a new concept, we shall consider with the author a table *coalescent*, when both the argument and the tabular function are separated into several sets of figures and these sets are located in various parts of the page, leaving it to the user of the table to collect the sets and to annex them to each other in the proper manner. The author refers to several publications in which he described his method of constructing coalescent tables, with the latest previous application to multiplication tables, and now extends his method to tables of squares and cubes.

The folding plate, accompanying the article, contains outlines of seven tables. Tables (A), (B), and (C) give three forms of coalescent tables for squares of 5-digit numbers. Table (D), consisting of two subdivisions, D(a) and D(b), with an auxiliary table (G), furnishes an outline of a table for squares of 6-digit numbers. Table (H) deals with cubes of 4-digit numbers.

In order to give an idea of the method pursued by the author we shall confine ourselves to the description of T. (A), this being the simplest in form. It furnishes the headings and the numerical results of the first two lines of page 1 of his table, but as his page is too wide for *MTAC*, we shall print his line of headings in two parts:

part I	00 | 1 6 — 3 8	2 7 — 2 7	3 8 — 1 6
part II	4 9 — 5	5 — 4 9	99

Each page thus consists of five vertical sections, corresponding to the first figure of the argument: (i) 1 or 6, (ii) 2 or 7, (iii) 3 or 8, (iv) 4 or 9, and (v) 5. (Each couple, 1 and 6, etc., is followed by a "dash," the purpose of which will be explained presently.) The 5-digit argument is divided into three sets, of 1, 2, and 2 figures resp. As we have already seen, the first figure is placed in the heading line. The second set (2d and 3d figures) of the argument are placed in the top corners: those running from 00 to 49 being in the left corner, and from 50 to 99 in the right corner. Thus the complete table will require 50 p., and p. 1 has 00 in the left top corner, and 99 in the right top corner. Finally, the third set (the 4th and 5th

figures) of the argument are contained in the first and last columns, 01 to 99, and 99 to 01 respectively.

Here we reproduce the first and fifth sections of the table, with the two horizontal lines of figures, given by the author in the outline:

00	1	6	—	3	8		5	—	4	9	99
01	1000	36001	20001	15999	80998		25001	00001	24999	99998	99
02	1000	36002	40004	15998	80996		25002	09004*	24998	99996	98

* erratum for 00004

While the 5-digit argument is separated into 3 sets, its 9 or 10-digit square is separated into 2 sets only, (1) the right 5 figures and (2) the left 4 or 5 figures. The right 5 figures are entered in the middle column under the "dash"; while the left 4 or 5 figures are entered in the column headed by the first figure of the argument, to the left or to the right of the middle col., depending upon whether the 2d set of the argument, in the top corner, is to the left or to the right.

For instance,

$$1\,0001^2 = 1000\ 20001 \qquad 3\,9999^2 = 15999\ 20001$$
$$6\,0001^2 = 36001\ 20001 \qquad 3\,9998^2 = 15998\ 40004$$
$$6\,0002^2 = 36002\ 40004 \qquad 4\,9998^2 = 24998\ 00004.$$

For the other tables, (B) to (H), which are more complicated, there is an additional feature: the author uses in one and the same table figures of different sizes, as ordinary, large and small, and of different types, as ordinary, bold and italics. In the latter two classes, a bold figure requires the increase of a certain figure in the preceding set by 1, and an italic figure a similar increase by 2. In view of this, one can easily understand how difficult it is for the printer and the proof reader to produce a table of that kind without errors, and the subsequent difficulty for the user of the table to obtain the correct result from the table.

As an illustration, the following is a list of errata found by the reviewer while reading the art., without an attempt to make the list complete. Some of the misprints appear in the Russian edition only, some in the English only, and some in both.

T. (A), section 5, col. "dash," l. 02, *for* 09004, *read* 00004	(R. and E.)
T. (B), last sec., col. 2, l. 1.6, *for* 152, *read* 153	(R. and E.)
T. (B), last sec., col. 8, l. 3.8, *for* 716, *read* 816	(E.)
T. (B), last sec., col. 1, l. 4.9, *for* 990, *read* 990	(R. and E.)
T. (B), last sec., col. 9, l. 4.9, *for* 655, *read* 655	(R. and E.)
T. (C), last sec., col. 6, l. 09, *for* 9126, *read* 9216	(R.)
T. (Da), last sec., col. 1, l. 4.9, *for* 9950, *read* 9950	(R. and E.)
T. (Da), last sec., col. 7, l. 4.9, *for* 4901, *read* 4901	(R. and E.)
T. (Db), col. 4, l. 6, *for* 2329, *read* 2349	(E.)
T. (Db), col. 6, l. 3, *for* 3336, *read* 4336	(R.)
T. (H), col. 5, l. 9, *for* 500779, *read* 500749	(R.)

It may be well to conclude this review by pointing out defects in the English text, which are due to faulty translation from the original Russian. The word *cipher* does not have the accepted meaning of *zero* (0), but is a translation of the Russian word *tsifra* (French *chiffre*) and means figure, digit, numeral. A second instance, perhaps even more serious, due either to a hasty reading or to misunderstanding of the Russian text, is the following: p. 464, sec. 2, l. 2, *for* the first, or the second, or the third cipher, *read* the first, second and third figures; similarly, l. 3, 4, *for* the fourth, or the second, or the third cipher, *read* the fourth, second and third figures.

S. A. J.

339[A, B].—NBSMTP, *Tables of Fractional Powers*. New York, Columbia University Press, 1946, xxx, 486 p. 18.6 × 26.5 cm. Reproduced by a photo-offset process. $7.50.

This new volume, produced by the prolific computing staff of the NBSMTP, under the technical directorship of Dr. A. N. LOWAN, is destined to be one of the most useful yet produced by this group. Nearly 500 of its pages are devoted to tabulations of the two functions A^x and x^a over ranges of the variable x for a selected set of values of A and a. The foreword is written by F. BERNSTEIN of New York University who makes the following pertinent observations: "The table of fractional powers presented here for the first time may be regarded, in a sense, as a generalization of BARLOW's tables, and it would be impossible to enumerate the many fields in which the present tables will be found useful. Suffice it to say that the algebraic and even the transcendental counterpart of numerous linear problems can now be attacked numerically with the prospect of success."

The first part of the volume, some 278 pages, contains five tables which give 15D values of the function A^x for the following values of A and x:

Table 1. $A = 2(1)9$ and $x = .001(.001).01(.01).99$.

Table 2. $A = 10$, and $x = .001(.001)1$.

Table 3. $A = \pi$, and $x = .001(.001)1$. The following special values of x are also included: $\pm x = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$, and $1(1)12$.

Table 4. $A = .01(.01).99$, and $x = .001(.001).01(.01).99$.

Table 5. $A = .001\ P$, where P is any prime between 100 and 1000, and $x = .001(.001).01(.01).99$.

The second part of the volume, some 208 pages, is devoted to six additional tables which give the values of x^a over the ranges indicated below. The first five tables, **Table 6** to **Table 10** inclusive, are computed to 15D, but the last, **Table 11**, is to 7D. The ranges over which the tabulations are made are as follows:

Table 6. $a = \pm \frac{1}{2}$ and $x = 0(.01)9.99$.

Table 7. $a = \pm \frac{1}{3}$ and $x = 0(.01)10$.

Table 8. $a = \pm \frac{2}{3}$ and $x = 0(.01)10$.

Table 9. $a = \pm \frac{1}{4}$ and $x = 0(.01)9.99$.

Table 10. $a = \pm \frac{3}{4}$ and $x = 0(.01)9.99$.

Table 11. $a = .01(.01).99$ and $x = 0(.01).99$. Values are also included over a variable range: $x = .001(.001)m$, where m never exceeds .3.

It will appear from this description that we have in this volume a set of tables which may be adapted for a wide range of uses. In the first place the familiar square and cube roots, together with their reciprocals, are found in Tables 6 and 7. However, these are not likely to replace the familiar BARLOW-COMRIE tables, except where more than nine significant figures are required, or where reciprocals are needed. The latter include square roots of x and $10x$ over the range $x = 1(1)12500$, a total of 25000 entries. The new tables give only a thousand square roots.

The present work may also be used as a brief table of antilogarithms, since Table 2 gives the values of 10^x. Here again only a thousand values are recorded, whereas in the classical tables of J. DODSON 100000 entries are given. But to quote the introduction: "Some similar tables with a smaller interval between the arguments are available, but the best of them are scarce."

One is pleased to find in the present volume tables of π^x over a generous range of values of x. Powers of π, integral, reciprocal, and fractional, have been computed in the past in some abundance and to high decimal approximation, but for the most part these are scattered in many places and are difficult to find. Hence it is very pleasing to have these values now readily accessible.

Interpolation in the present tables will be found to be difficult, since no differences are given. It is clear, however, that when values are tabulated to 15D at interval .01 a complete differencing would have required the printing of differences of high order. Nevertheless the publication of first differences would have served a useful purpose since the error in linear

interpolation for A^x will be less than $.7h^2 (\log A)^2$, where h is the tabular interval. Throughout most of the values of A the accuracy thus achieved is from 4 to 6D. Quadratic interpolation by means of the 3-point Lagrangean interpolation coefficients has an error less than $.8h^3 (\log A)^3$. Throughout most of the range of A, quadratic interpolation would thus give an accuracy varying from 6 to 9D. Similar formulae are given for the errors of linear and quadratic interpolation in the tables of x^n.

The methods of computation and the means to insure accuracy are described in the Introduction. Thus the construction of the values given in Part I was achieved from the key values $A^{0.001}$, which were computed to 20 places by means of the formula,

$$A^x = 1 + \frac{x \ln A}{1!} + \frac{(x \ln A)^2}{2!} + \frac{(x \ln A)^3}{3!} + \cdots,$$

where $\ln A$ was taken from WOLFRAM's table. For this computation seven terms of the series were found sufficient. Powers of the key values yielded the desired tabular entries.

In the computation of cube roots a method of successive approximation was used by means of the formula: $x^{\frac{1}{3}} = p_1 + h$, where p_1 is the 6D value taken from BARLOW's *Tables*. An estimate of h is obtained from the expansion $x = p_1^3 + 3p_1^2 h + \cdots$, and the corrected value again employed in place of p_1. It may be of interest to observe that the author of this review, in making similar calculations, discovered that more rapid progress could be made by computing $y = \frac{1}{3} \ln x$ and then evaluating $x^{\frac{1}{3}}$ by means of the NBSMTP tables of e^y. The difficulty of inversion from existing tables of common logarithms to more than 10D, and the lack of antilogarithm tables to 15D, preclude the use of the more usual method of computation.

The volume under review also provides an extensive bibliography of tables of fractional powers. Seventy-six contributions to the subject are listed.

H. T. D.

340[B].—LUDWIG ZIMMERMANN, *Vollständige Tafeln der Quadrate aller Zahlen bis 100009*, third edition. Berlin, Wichmann, 1938. Lithoprinted by Edwards Bros., Ann Arbor, Mich., 1946, xix + 187 p. 19 × 25 cm. $2.80. Compare MTE **89**, *MTAC*, v. 2, p. 180, note on p. 444; the first and third editions are practically identical.

This table is the largest table of squares in print. As the title indicates, the main table gives the exact values of the first 100009 squares. The table is similar in its arrangement to the usual table of 7-place logarithms, the final digits of the 5-figure arguments being used as column headings. Eight digits are given in the body of the table; the leading two digits are printed in the upper left hand corner of the page and in the left column when these digits change. The extreme right column is headed d and is to be used in computing the difference of the tabulated squares. In fact the difference $\Delta(N^2)$ is merely $1 + 2t + 10d$, where t is the last digit of N.

This table should not be used for finding squares of isolated 5-digit numbers, unless one has no computing machine. Depending on the type of machine, the computer can calculate and copy down the square of such a number in from one half to three quarters of the time required to read it out of the table. The reviewer has found the table quite handy as an auxiliary to a desk calculator in finding square roots to 10 significant figures. The first five figures of $N^{\frac{1}{2}}$ can be read from the table and the last five can be read from the machine while it automatically divides N by this approximation to $N^{\frac{1}{2}}$.

To the computer who uses no calculating machine the table is of course a great help in finding squares, square roots, and products (by the method of quarter squares). Discussing the latter use of the table the author asserts that, by using this table together with a 5-figure natural trigonometric table, problems in trigonometry may be solved more easily than by using 6-place logarithms. The examples in the introduction all suppose that the user is without a calculating machine.

When one considers the rather crowded printing on each page and the fact that many such users may not need the full ten digits of a square one wonders whether possibly another arrangement suggested by the rather obscure table of GOSSART[1] might not be preferable. In this arrangement the first (rather than the last) digits of the 5-figure argument would be used as column headings. Thus page 154 might be laid out in larger type somewhat as follows:

	1 3	2 8	3 14	4 23	5 34	6 46	7 61	8 78	9 97	
8500	4225	1225	8225	5225	2225	9225	6225	3225	0225	0000
8501	4228	1230	8232	5234	2236	9238	6240	3242	0244	7001
8502	4232	1236	8240	5244	2248	9252	6256	3260	0264	4004
....	....	....	....	....	....	....	....	....	....	....
8549	4406	1504	8602	5700	2798	9896	6994	4092	1190	5401

Thus the first six significant figures of the square of 68502 are 469252. If the exact value of this square is needed one merely annexes the four digits in the extreme right column to obtain

$$68502^2 = 4692524004.$$

This arrangement would require about 2250 printed figures per page instead of the 4200 actually used.

There are two other two-page tables as follows:

Table II (p. 184–185) giving the first 1000 cubes.

Table III (p. 186–187) giving the first hundred k-th powers for $k = 4(1)9$. This table contains the following two errata[2] unfortunately carried over into the present edition: 27^9, *for* 7625597484387, *read* 7625597484987; 77^6, *for* 208422390890, *read* 208422380089. The name Zimmermann is misspelled on the back of the volume. The quality of the lithoprinting is high. Every entry is clear and uniform in spite of the heavy character of the page.

D. H. L.

[1] ALEXANDRE GOSSART, *Table des Carrés de 1 à 100 Millions* . . . Paris, 1865, 86 p. There are also copies of this book with the title beginning *Table des Carrés de un à cent Millions*.

[2] These errata are given by J. C. P. MILLER, in BAASMTC, *Mathematical Tables*, v. 9, *Table of Powers*. Cambridge 1940, p. xii.

341[B, C, D].—VÁCLAV ELZNIC & MILOSLAV VALOUCH, *Geoma. Pětimístné Tabulky hodnot a Logaritmů goniometrických funkcí v setinném dělení, Tabulky geodetické a katastrální* [Five-place tables of values and logarithms of trigonometric functions in the centesimal angle division, geodesic, and surveying tables], Prague, Czechoslovakia, Jednota Českých Matematiků a Fysikû [Society of Czech Mathematicians and Physicists], 1944. 319 p. 16.7 × 25.4 cm. Paper bound 178 Czech crowns.

Before the recent war this Czech Society, which was founded nearly 85 years ago, was the most affluent mathematical society in the world, and also one of the very largest. It owned its own building and printing presses, and did an enormous publication business especially in texts used in schools throughout Czechoslovakia. Besides various mathematical periodicals published during the past 75 years it sent forth a score of treatises on advanced topics in the fields of mathematics and physics.

The main tables of the present volume for geodesists and surveyors are natural trigonometric functions with centesimal divisions of the quadrant as argument, the use of which beginning late in the eighteenth century has been already set forth in *MTAC*, v. 1, p. 33–39. This centesimal unit has made no headway in America, but was widely used in European countries during the recent war, for example, J. T. PETERS, *Sechsstellige Werte d. trigonometrischen Funktionen von Tausendstel zu Tausendstel des Neugrades*, of which there were at least 7 editions, 1938–1943.

The tables in the volume under review are as follows:

P. 11, $x = a \cot a$(6S), and cot a(4S), for 213 values of a from 0 to $2^g.009$.
P. 12–27, cot a, $a = [2^g(0^g.001)9^g; 6S]$, Δ, p.p.
P. 28–127, sin a, tan a, cot a, cos a, K, for $a = [0(0^g.01)50^g; 5D]$, Δ, p.p., where $K = 1 + \sin a + \cos a$.
P. 128–147, N^2 for $N = [0(.001)10.009; 4D]$, p.p.
P. 148–169, log N, $N = [0(1)11009; 5D]$, p.p. Also S, T, log sin a, log tan a, $a = [0(0^g.01)1^g.1; 5D]$.
P. 170–181, log sin a, log tan a, log K, $a = [0(0^g.001)3^g; 5D]$, Δ.
P. 182–275, log sin a, log tan a, log cot a, log cos a, log K, $a = [3^g(0^g.01)50^g; 5D]$, Δ, p.p.
P. 276, conversion tables of degrees to radians and radians to degrees.
P. 277, tables for centesimal measure $\sin \epsilon/(\sin 1'' \cdot 10^6)$.
P. 278–281, tables and geodesic constants of the Bessel and international ellipsoids.
P. 282–284, tables and examples for the solutions of the principal geodesic problems.
P. 285, tables for correction of the curvature of the earth and refraction in trigonometric measurement of heights.
P. 286, interpolation coefficients for Newton's formula, Δ.
P. 287, interpolation coefficients for Bessel's formula, Δ, with second difference table.
P. 288–292, formulae in trigonometry.
P. 292–294, series, constants, interpolation.
P. 295–297, measures of length in various countries, surfaces, areas, etc.
P. 298, constants and a table of arcs subtended by central angles $a = [0(10')10^g; 10D]$. In the table of constants the thirtieth decimal place in the value for π^{-2} should be increased by unity. There are also several errors on p. 4, lines 6, 10, 12, and −4.

R. C. A.

342[B, C, F].—R. DE MARCHIN, *Tables Numériques, à l'usage de l'enseignement moyen et normal, des Écoles Industrielles, des candidats aux Écoles Spéciales des Universités*. Brussels, A. Manteau, 1942. ii, 36 p. 16 × 24.2 cm.

This little volume by a professor in the Athénée at Herstal, near Liége, is intended primarily for use in places indicated by the title. It presents some tables not often found in such booklets of other countries. The tables are as follows:

1. Log N, p. 6–9, 5D table, with p.p., for $N = 100(.1)200(1)999$. On p. 10 there is a serious error; the values of N, 119 and 118 are interchanged. Ln N, p. 10–15, 5D table, with p.p., for $N = 100(.1)400(1)999$.
2. N^2, $N = 1(1)999$, p. 16–17.
3. Tables for finding N^2, $N < 10^6$, p. 18.
4. N^4, N^4, for $N = [1(1)100; 10D]$, p. 19.
5. N^3, $N = 1(1)999$, p. 20–21.
6. N^p, $p = 4(1)9$, $N = 1(1)50$; $p = 4(1)8$, $N = 51(1)99$, p. 22–24.
7. Inverse table of Euler's totient function $\phi(N)$ giving solutions x of $\phi(x) = N$ for $N \leqslant 100$, p. 25. The author states that one may consult discussions of this "Gauss" function in
   PHILIPPENS-VAN DE WERKE, *Arithmétique*, ninth ed., Namur, Wesmael, 1939, p. 181.
   HERBIET-HORWART, *Traité d'Arithmétique*, Namur, Wesmael, 1933, p. 205, 216.
   HORWART, *Questions d'Arithmologie*, Namur, Wesmael, 1940, p. 212.
   SCHONS, *Traité d'Arithmétique*, Namur, La Procure, 1938, p. 236.
   GOSS, *Traité d'Arithmétique*, Tongres, Michiels, 1934, p. 141.
8. Table of the number of digits in the periodic and (or) nonperiodic parts of the decimal expansions of the reciprocals of $N = 2(1)50$, p. 25.

9. Table of prime numbers <100110, p. 26–33.
At the bottom of each of seven of the pages is the statement of a theorem in the theory of numbers, for example:

(a) If the prime number p does not divide a, it divides $a^{p-1} - 1$ (Fermat)
(b) If p is prime to a, and if a^{p-1} is the least power of a which equals a multiple of p plus 1, p is prime. (Lucas)
(c) For p to be a prime it is necessary and sufficient that the sum $(p-1)! + 1$ be divisible by p. (Wilson)
(d) If a and b are prime to one another, b divides the expression $a^{\phi(b)} - 1$, ϕ defined as above. (Euler)

10. Frequency tables of the 9592 prime numbers $<10^5$, (a) by hundreds; (b) by thousands. In (c) is a table of the number of hundreds containing n primes, n varying from 1 hundred containing 25 primes down to 3 hundreds containing 3 primes. p. 34.
11. Table of the composite numbers $<10^4$ whose smallest divisor (13 to 97) is >11, p. 35–36.
12. Table of primitive rational right triangles $a^2 = b^2 + c^2$, arranged according to $a < 1000$, p. 36. The author adds, for properties of the sides of a primitive triangle consult:

HORWART, *Questions d'Arithmologie*, Namur, Wesmael, 1940, p. 125.
HERBIET-HORWART, *Traité d'Arithmétique*, Namur, Wesmael, 1933, p. 193–194.
SCHONS, *Traité d'Arithmétique*, Namur, La Procure, 1938, p. 225.

R. C. A.

343[D].—CH. FÉVROT, "Caractéristique théorique d'un projecteur à miroir parabolique à source cylindrique axiale," *Revue d'Optique*, v. 23, Oct.–Dec. 1944 (publ. Apr. 1946), p. 264–267. 15.9 × 24 cm.

We have here a table and supplementary graphs. The table (p. 266) gives the values of $2 \sin^{-1} [2k(1 + m^2)]^{-1}$ in degrees and fractions of a degree (tenths or hundredths), for $m = .1(.1)1(.2)3$, and for $k = .05(.025).1(.05).3(.1)1(.2)1.6$. [The second argument $k = .15$ is incorrect and should be .5; furthermore, under $k = .15$, 180° opposite $m = 5$ is also incorrect.] The table is completed by figures (p. 264, 265, 267), which do not give the angle but $m \times 2 \sin^{-1} [2k(1 + m^2)]^{-1}$, as functions of m for the different k.

R. C. A.

344[D, R].—VÁCLAV ELZNIC, (a) *Sintacos 10. Desetimístné Tabulky hodnot goniometrických funkcí sin, tg, cos, pro setinné dělení kvadrantu*, [Sintacos 10. Ten-place tables of values of the trigonometric functions sin, tan, cos, for centesimal division of the quadrant]. Prague, 1941, 112 p. 15.5 × 22.6 cm. Paper covers 98 Czech crowns.
(b) *Osmimístné Tabulky, přirozených hodnot goniometrických funkcí sin, cos, tg a Tabulky Geodetické pro úhlové dělení šedesátinné.* [Eight-place tables of values of the trigonometric functions sin, cos, tan, and geodetic tables, for sexagesimal angle division.] Prague, 1940, 64 p. + interpolation table card, and errata slip. 18.8 × 27 cm. Boards, 54 Czech crowns. These volumes are publications of the Jednota Českých Matematiků a Fysiků [Society of Czech Mathematicians and Physicists]. Compare RMT **341**.

(a) The main table, p. 6–105 is of sin x, cos x, for $x = [0(0^g.01)100^g$; 10D], Δ^2; also tan x, 0–50^g, cot x, 50^g–100^g, each at interval $0^g.01$, 10D, Δ^2. On p. 106–112 are reprints of Andoyer's 20D tables of sin x and cos x, each with v^8, and tan x with v^{12}, $x = 0(1^g)50^g$. Of previous tables of this type and interval was the 8D table, Δ, of H. C. C. ROUSSILHE & BRANDICOURT, *Tables à 8 Décimales des Valeurs naturelles des Sinus, Cosinus, Tangentes . . .*, new

ed., Paris, 1933. But no previous printed table of greater than 8D was as extensive throughout the quadrant as the one under review.

(b) In this v. **T. II**, the principal table, p. 7–51, is of sin x, tan x, cos x, and $K = 1 + \sin x + \cos x$, for $x = [0(1')45°; 8D]$, Δ^2, with columns for $\Delta 1''$ and $\Delta 1'$ for each entry. Each page of this table contains at the left a table of the variations in direction per km., a^*, b^*, with Δ.

T. I, p. 6, is of cos x, for $x = [0(10'')1°; 10D]$, Δ^2.

T. III, p. 52, is a 15D conversion table for arcs into radians, $x = 0(1°)120°$; $0(1')60'$; $0(1'')60''$.

T. IV–V, p. 53–54, are for conversion of (i) sexagesimal units into time, (ii) degrees into centesimal units, and centesimal units into sexagesimal.

T. VI–VIII, p. 55–60, geodesic; **T. IX–XI**, p. 60–63, cartographic; **T. XII** orthometric projection; **T. XIII** terrestrial curvature and refraction; **T. XIV** spherical excess; **T. XV–XVI** pendulum; **T. XVII** variation in attractions of the earth for varying altitudes. On one side of the card, inserted loose in the v., is a table, **App. A** for second difference correction; on the other side, **App. B**, a table of $\Delta/60$, for $\Delta = [0(.1)60; 5D]$.

In the table of constants on p. 5 there are three errors; the thirtieth decimal places of the values for π, M and e should each be increased by a unit.

R. C. A.

345[F].—Lord Cherwell, "Note on the distribution of the intervals between prime numbers," *Quart. Jn. Math.*, Oxford s., v. 17, 1946, p. 46–62.

This article gives data on the distribution of pairs of primes differing by a given amount. These data are compared with approximate formulae derived by probability methods. In particular, **Table 2** gives the number of primes p with $250000 < p < 255000$ for which $p + k$ is a prime for $k = 2(2)98$. **Table 4** gives the number of primes whose difference is k, in the ranges 500–2000, 2000–10000, 10000–30000, 90000–110000, 220000–255000 with $k = 2(2)62$. Glaisher's[1] counts of the twin primes in the first 100000 numbers of the n-th million for $n = 2, 3, 7, 8, 9$ are reproduced in **Table 1(a)**. There are also data on the distribution of triplets of primes p, $p + 2$, $p + 6$ and p, $p + 4$, $p + 6$ for the 5 ranges mentioned above and also for $m \cdot 10^5 < p < (m + 1) \cdot 10^5$ with $m = 1(1)9$.

The author was apparently unaware of a paper by Sutton[2] (also employing a probability argument) giving data on the distribution of twin primes up to 800,000. Unfortunately the two sets of data do not agree in the few places where they are comparable. Thus Table 4 gives 198 as the number of twin primes between 90000 and 110000 whereas the correct number is 204, as given by Sutton. The probability of missing a few primes in such a count is not small, as the reviewer has found by very little experience, and errors of omission are apt to be about as large as the discrepancies between the actual count and the conjectured approximate formulae being tested.

D. H. L.

[1] J. W. L. Glaisher, "An enumeration of prime-pairs," *Mess. Math.*, v. 8, p. 28–33, 1878. See also BAAS, *Report, 1878*, p. 470–471.

[2] C. S. Sutton, "An investigation of the average distribution of twin prime numbers," *Jn. Math. Physics*, v. 16, 1937, p. 1–42.

346[F].—Albert Delfeld, "Table des solutions de la congruence $X^4 + 1 \equiv 0 \pmod{p}$ pour $300000 < p < 350000$," Institut Grand-ducal Luxembourg, Section des Sciences, *Archives*, v. 16, 1946, p. 65–70.

This table is an extension of three previous tables of Cunningham[1] ($p < 10^5$), Hoppenot[2] ($10^5 < p < 2 \cdot 10^5$) and Gloden[3] ($2 \cdot 10^5 < p < 3 \cdot 10^5$), all giving the two least positive solutions of the congruence $X^4 \equiv -1 \pmod{p}$.

The method of solution, based on the two quadratic partitions of p: $p = x^2 + y^2 = 2z^2 + t^2$, is described in *MTAC*, v. 2, p. 71. This table contains entries corresponding to

the 980 primes p of the form $8n + 1$ between the limits $3 \cdot 10^8$ and $3.5 \cdot 10^8$. The author refers to a manuscript table by Gloden for $3.5 \cdot 10^8 < p < 5 \cdot 10^8$. No doubt this upper limit is a misprint for $5 \cdot 10^8$.

D. H. L.

[1] A. J. C. CUNNINGHAM, *Binomial Factorisations*, v. 1. London, 1923, p. 23–33; v. 4, London, 1923, p. 19–37.

[2] S. HOPPENOT, *Table des Solutions de la Congruence* $X^4 \equiv -1 \pmod N$ *pour* $100000 < N < 200000$, (Librairie du Sphinx). Brussels, 1935, 18 p.

[3] A. GLODEN, "Table des solutions de la congruence $X^4 + 1 \equiv 0 \pmod p$ pour $2 \cdot 10^8 < p < 3 \cdot 10^8$," *Mathematica* (Timişoara), v. 21, 1945, p. 45–65.

347[F].—EDWARD B. ESCOTT, "Amicable numbers," *Scripta Math.*, v. 12, 1946, p. 61–72. 16.8 × 24.6 cm.

This is a table described in UMT 9, *MTAC*, v. 1, p. 95–96, with introductory text, partially quoted from Mr. Escott's *MTAC* note.

348[F].—A. GLODEN, "Table de factorisation des nombres $X^4 + 1$ dans l'intervalle $1000 < x \leqslant 3000$," Institut Grand-ducal Luxembourg, *Section des Sciences, Archives*, v. 16, 1946, p. 71–88.

This table is a by-product of tables of solutions of the congruence $x^4 + 1 \equiv 0 \pmod p$, referred to in RMT **346**, which extend (if one includes a manuscript table of the author) to $p < 500000$. Although this upper limit serves to factorize all numbers $x^4 + 1$ only up to $x = 708$ when x is even and $x = 833$ when x is odd, nevertheless a large number of numbers $x^4 + 1$ beyond this range have relatively small factors and hence succumb to this method. The present table for $1000 < x \leqslant 3000$ is an extension of a previous table of CUNNINGHAM.[1] Of its 2000 entries, 1417 are completely factorized, 124 partially factorized, and the remaining 459 are of unknown character. All unknown factors exceed 500000. In order to discover more of these factors the author announces his intention of extending the congruence table to $p < 650000$. One eventually reaches a point of diminishing returns by this method since it becomes easier to attack each number to be factored directly. Just where this point is, depends upon the factorization methods at one's disposal. The table contains only one prime entry: $\frac{1}{2}(1089^4 + 1) = \frac{1}{2}(33^8 + 1) = 703204309121$, due to N. G. W. H. BEEGER.

D. H. L.

[1] A. J. C. CUNNINGHAM, *Binomial Factorisations*, v. 1. London, 1923, p. 113–119. See also p. 139 for 38 cases of $y^4 + 1$ with $y > 1000$. Erratum: $y = 2518$, *for* 461801, *read* 4681801.

349[G].—H. DAVENPORT, "The product of n homogeneous linear forms," Akad. v. Wetens., Amsterdam, *Proc.*, v. 49, 1946, p. 827. 18.1 × 26 cm.

$x^{-a} + (x - 1)^{-a} = 1$, $f(a) = (ax + x^{-a})/(a + 1)$. There are tables of x, x^{-a}, $f(a)$, for $a = [.5(.1)3; 3D]$.

350[I, O].—I. J. SCHOENBERG, "Contributions to the problem of approximation of equidistant data by analytic functions. Part B. On the problem of osculatory interpolation. A second class of analytic approximation formulae," *Quart. Appl. Math.*, v. 4, July, 1946, p. 112–141. The tables, p. 136–141, are by Mrs. MILDRED YOUNG. 17.7 × 25.4 cm. See *MTAC*, v. 2, p. 167–169, for a review of Part A; also Corrigenda p. 228.

The numbering of the formulae in this review continues that of the previous one. Schoenberg's notation D^m, C^k, E^k, s will be used to designate a broken polynomial base

function of degree m, with μ continuous derivatives, which reproduces every polynomial of degree $\leqslant k$ exactly, and is of span s.

In this paper Schoenberg extends his theory to the case where his polynomial base functions are not spline functions, but have some lower order of continuity. As before, there are two main parts: a first part deals with the broken polynomials themselves, while the second deals with analytic base functions derived from them by considerations based on heat flow. These two chapters are preceded by one containing some lemmas whose motivation is not clear until one has got far into the paper.

The underlying motif of the first part of the paper may be described as follows. If $g(u)$ is the characteristic function (in the sense of Fourier analysis) of a base function $L(x)$ satisfying Schoenberg's restrictions, then Schoenberg proved in Part A that a necessary and sufficient condition that $L(x)$ "preserves the degree $k-1$" [i.e. transforms every polynomial of degree $\leqslant (k-1)$ into one of the same degree with the same leading coefficient] is that $g(0) = 1$ and for every integer $n \neq 0$, $g(u)$ has a zero of order at least k at $u = 2\pi n$. Then if $\psi_k(u)$ is defined by (10), with $t = 0$, it follows that

$$g(u) = \omega(u)\psi_k(u), \tag{17}$$

where $\omega(u)$ is regular in a horizontal strip containing the real axis. Special interest attaches to the case where $L(x)$ is an "ordinary" interpolation formula. There are three types of these which Schoenberg specially considers.

(1) (Schoenberg's second type). Here $\omega(u)$ is periodic (with period 2π). In that case $\omega(u) = [\phi_k(u)]^{-1}$, where ϕ_k is defined by (11) with $t = 0$. The resulting interpolation formula is D^{k-1}, C^{k-2}, E^{k-1} but is of infinite span if $k \geqslant 3$. This is a spline; compare (12).

(2) (Schoenberg's first type). Here

$$\omega(u) = \omega_1(u) + \omega_2(u)\psi_{k+1}(u)/\psi_k(u) \tag{18}$$

where $\omega_1(u)$ is of period 2π. Instead of taking ω_2 with period 2π, Schoenberg shows that there exist even functions ω_1 and ω_2 with $\omega_1(u + 2\pi) = \omega_1(u)$, $\omega_2(u + 2\pi) = -\omega_2(u)$, such that the resulting $L(x)$ is D^k, C^{k-2}, E^{k-1}, and $s = 2k - 2$ if k is even and $2k - 1$ if k is odd. This settles a conjecture of Greville's as to the existence of ordinary interpolation formulae of such character.

(3) (Schoenberg's third type). Here $\omega(u)$ is a polynomial in u^2 consisting of the Maclaurin expansion of $[\phi_k(u)]^{-1}$ as far as the term in $u^{2\mu-2}$, where μ = the integral part of $\frac{1}{2}(k+1)$. The resulting formula is the k-point central difference (i.e. Lagrangean) formula.

Of these three types the first and third are useless as ordinary formulae, but they give rise to smoothing formulae by truncating the $\omega(u)$. Thus in the first case it is shown that $\omega(u)$ can be expanded as a power series in $s = \sin^2 \frac{1}{2}u$ with positive coefficients. If the first $m-1$ terms of this series are taken, where $0 < 2m < k+2$, we get a formula, designated $L_{k,m}(x)$ which is D^{k-1}, C^{k-2}, E^p, $s = k + 2m - 2$, where $p = \min(2m-1, k-1)$. In the third type if the Maclaurin series is stopped with the term in u^{2m-2} with $m < \mu$ we get a smoothing formula of type D^{k-1}, C^{k-2m}, E^{2m-1}, $s = k$.

In the second part of the paper Schoenberg considers formulae $L_{k,m}(x, t)$ which are derived from the $\psi_k(x, t)$ given by (10) in the same way as the $L_{k,m}(x)$ were derived from $\psi_k(x)$ above. It is shown that $L_{k,m}(x, t)$ is an analytic interpolation formula which is exact for the degree min $(2m-1, k-1)$ and preserves the degree $k-1$. The smoothing properties of $L_{k,m}(x, t)$ are, however, frankly left to depend on a conjecture. It is stated that $L_{k,m}$ damps out as $x \to \infty$ like $e^{-c^2x^2}$; whereas the L_k in Part A damps out only like e^{-c^2x}.

In the tables three functions $L_{4,2}(x, \frac{1}{4})$, $L_{4,2}(x, \frac{1}{2})$, and $L_{4,2}(x, \frac{3}{4})$ are tabulated to 8D, together with their first and second derivatives to 7D. The interval in x is .1, and the tables are arranged so that the values for x differing by an integer are in the same column. This facilitates subtabulation. The range of the tables varies; it includes the entire range where the function concerned is not zero to 8D—the approximate average is from -5 to $+5$.

As the author states explicitly the tables are only examples to illustrate the method. For practical use, with varying requirements as to smoothness, accuracy, etc., a more complete set of tables would be necessary.

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351[K, L, M].—A. J. BARKLEY ROSSER, *Theory and Application of* $\int_0^z e^{-z^2}\,dx$ *and* $\int_0^z e^{-x^2y^2}\,dy \int_0^y e^{-z^2}\,dx$. *Part I. Methods of Computation. Final Report* OSRD 5861, series B, no. 2.1 from the Alleghany Ballistics Laboratory, division 3, section H, Office of Scientific Research and Development, National Defense Research Committee, xii, 212 p. Nov. 1945. Offset print. 20.1 × 26.6 cm. **B.** EDWARD M. COOK & J. B. ROSSER, *Basic Computational Data* used at ABL in the *Preparation of the Numerical Tables in ABL Final Report, OSRD no. 5861, Part I.* 44 leaves, offset print on one side of each leaf. Nov. 1945. 20.1 × 26.6 cm. These publications are available only to certain Government agencies and activities.

A. The need for numerical values of the error function $\int_0^z e^{-z^2}dx$ for complex values of z arises in many scientific fields, especially statistics, optics, heat conduction, electro-chemical diffusion, and the propagation of electromagnetic waves. The present report consists mainly of a collection and extensive development of the known methods for calculating both the error function, i.e. the single integral, and the double integral $G(z, p) = \int_0^z e^{-x^2y^2}dy \int_0^y e^{-z^2}dx$, where z and p are any two complex numbers. At the end, tables are given for auxiliary functions to aid in the computations of both the single and double integral.

The first 105 pages treat the single integral $\int_0^z e^{-z^2}dx$ (which is called either "error function" or "error integral"), or alternatively, the sometimes more convenient function $F(z) \equiv e^{z^2}\int_z^\infty e^{-z^2}dx$, and these pages comprise the more fundamental part of the report. The introduction establishes the notation and more obvious properties of the error function, (e.g. such as are seen by completing the square) and gives four other forms of $F(z)$. Then there follows a short discussion of the error function along the real, imaginary, or 45° rays, (the last being essentially Fresnel integrals), and the two direct power series expansions, i.e. the usual Maclaurin series and a series obtained by integrating by parts. In section 4. a number of fundamental inequalities are proved, in order to be used later, e.g. the simplest inequalities being those for z in the first quadrant, $|F(z)| \leqslant \frac{1}{2}\pi^{\frac{1}{2}}$, and for z^2 in the right half plane, $|F(z)| \leqslant 1/|2z|$. The next section, on computation by use of exponentials, is based upon Poisson's formula which expresses an infinite sum in terms of an infinite sum of cosine transforms. The equivalent of a special case of Poisson's formula is proved from first principles, and then it is applied to a function of exponential type, to obtain an equation between two infinite series of exponential terms. It is shown in several ways that it is possible to choose the parameters so that undesired terms, which usually include all of the left or right member of the equation, become negligible, and an expression for the error integral is obtainable as the sum of essentially exponential functions. In section 6. there is a discussion of the usual asymptotic series, in reciprocal powers, for $F(z)$, and upper bounds are found for the remainder in the more difficult case when z^2 is not in the first quadrant, (when z^2 is in the first quadrant, it is easily shown that the remainder is less in absolute value than the first term neglected). The section on the asymptotic series closes with a list of the first twenty terms of the series, as powers of $\frac{1}{2}z^{-2}$. Section 7. discusses properties of polynomials $A_N(z)$ and $B_N(z)$ which are the respective denominators and numerators of the Nth convergent in the continued fraction expansion for $F(z)$, and lists $A_N(z)$ and $B_N(z)$ as functions of z for $N = 1(1)16$. Here $A_N(z) = (-i)^N H_N(iz)$, where H_N denotes the Hermite polynomial of order N. Section 8. is devoted to the continued fraction for $F(z)$, namely $\dfrac{1}{2z+}\,\dfrac{2}{2z+}\,\dfrac{4}{2z+}\,\dfrac{6}{2z+}\cdots$, and a proof is given of the fundamental theorem $F(z) = \dfrac{B_N(z)}{A_N(z)} + \dfrac{1}{A_N(z)}\dfrac{d^N F(z)}{dz^N}$. A detailed dis-

cussion of the remainder in approximating $F(z)$ by $\frac{B_N(z)}{A_N(z)}$ leads to a number of practical upper bounds for $\left|\frac{d^N F(z)}{dz^N}\right|$, and to the theorem that the continued fraction converges to $F(z)$ for z^2 in the right half plane. (The authors suspect convergence everywhere except on the imaginary axis, but are unable to prove it.) More involved expressions for the remainder are found when z^2 is in the second quadrant, and the section closes with a note on Laplace's form of the continued fraction. The next three sections are concerned with general asymptotic formulae and two approximate quadratic equations for $F(z)$, which are admitted to be of limited use. The final recommendation regarding the single integral gives preference to the asymptotic series and continued fraction, both for their simplicity and accuracy.

The treatment of the single integral might have been enhanced by noting that much greater accuracy by means of the asymptotic series for $F(z)$ could have been obtained by expanding the remainder in a number of ways. Thus there is a method described in J. R. AIREY, "The 'converging factor' in asymptotic series and the calculation of Bessel, Laguerre and other functions," *Phil. Mag.* s. 7, v. 24, 1937, p. 521–552. Another method, which came to the reviewer's attention because it was used extensively by several members of the NBSMTP, is to write the integrand of the remainder as an exponential and to integrate by parts. Thirdly, the remainder appears to have a rather simple expression by employing for $\int_z^\infty e^{-z^2}x^{-m}dx$ a continued fraction that is given in a manuscript of H. T. DAVIS (available in microfilm) and which appears worthy of study. This continued fraction, which does not seem to be widely known, is

$$\int_z^\infty e^{-z^2}x^{-m}dx = \frac{e^{-z^2}}{z^m}\left[\frac{1}{2z+}\,\frac{m+1}{z+}\,\frac{2}{2z+}\,\frac{m+3}{z+}\,\frac{4}{2z+}\,\frac{m+5}{z+}\,\cdots\right].$$

The rest of the theoretical material, p. 106–173, is devoted to the double integral $G(z, p)$. For the special case when $p = i$, the methods used for the single integral are generalized. Thus sections 16. and 19.–23. concern themselves more specially with $p = i$ and apply methods similar to the power series, Poisson formula, asymptotic series, continued fraction, and approximate quadratic equation. For the more difficult case of a general p, there are developed, in sections 13.–17. and 24., six different types of series of the form

$$\psi + X\sum_{N=0}^{\infty} K_N Y^N \phi_N(Z),$$

where ψ, X, Y, Z are functions of z and p, the K_N's are constants and the ϕ_N's are various functions of Z. Furthermore, for each set of ϕ_N's it is possible to employ more than one set of values of the ψ, X, Y, Z and K_N's. The validity of these various series is for restricted sets of z and p, and the convenience extends to an even more limited range of z and p; but that is compensated by the number of different series from which to choose. Also, in section 18., a rather elaborate method is described for calculating $G(z, p)$, which is analogous to the Poisson formula approach in section 5. Many of the methods for computing $G(z, p)$ require that values of the single integral be available. In all these methods, scrupulous attention is devoted to the estimation of the accuracy, usually by supplying an upper bound for the remainder. As a by-product of the study of $G(z, p)$, means are provided for evaluating four different types of integrals, namely

$$\int_0^\infty \frac{e^{-x}dx}{(x+z^2)^{N+\frac{1}{2}}}, \quad \int_0^\infty \frac{e^{-x}dx}{(x+z^2)^{\frac{1}{2}}(x+2z^2)^{N+1}}, \quad \int_0^\infty \frac{e^{-x}dx}{(x+z^2)^{N+\frac{1}{2}}(x+2z^2)},$$

and

$$\int_0^w y^{2N}F(y, c)dy.$$

In the appendix, p. 174–212, there are tabulated the following functions:

Table 1: $Rr(u) = \frac{1}{2}\cos\frac{1}{2}\pi u^2 - \frac{1}{2}\sin\frac{1}{2}\pi u^2 + C(u)\sin\frac{1}{2}\pi u^2 - S(u)\cos\frac{1}{2}\pi u^2$,
$Ri(u) = \frac{1}{2}\cos\frac{1}{2}\pi u^2 + \frac{1}{2}\sin\frac{1}{2}\pi u^2 - C(u)\cos\frac{1}{2}\pi u^2 - S(u)\sin\frac{1}{2}\pi u^2$,

(where $C(u)$ and $S(u)$ are the Fresnel integrals defined by

$$C(u) = \int_0^u \cos \tfrac{1}{2}\pi x^2 dx \quad \text{and} \quad S(u) = \int_0^u \sin \tfrac{1}{2}\pi x^2 dx,$$

$$Rr^2(u) + Ri^2(u), \quad \text{and} \quad \int_0^u Rr(x)dx.$$

All four of these functions are given to 12D, for $u = -.06(.02)3.5(.05)5$ and the first three are given also for $u = 5.05(.05)5.15$.

Table 2: $e^{-w^2}\int_0^w e^{y^2}dy$, $e^{w^2}\int_w^\infty e^{-y^2}dy$, $\int_0^w e^{-y^2}dy\int_0^y e^{x^2}dx$, and $\int_0^w e^{x^2}dy\int_y^\infty e^{-x^2}dx$.

All four functions are given to 10D. The first is for $w = -.2(.05)4(.1)6.5(.5)12.5$. The second is for $w = -.2(.05)3.8(.1)6.3$. The third and fourth are for $w = -.2(.05)+3.5(.1)6$. Sections 25. and 26. contain a discussion of the above mentioned tables which includes means of computation and checking. Section 27. discusses the method of interpolation, which is supplemented by the graphic representation of the error incurred in 2- to 6- or 7-point Lagrangean interpolation. At the end, 16 references are cited.

In evaluating this book, it is fitting to say that Dr. Rosser and his collaborators are to be highly commended for a rather thorough as well as rigorous investigation of the calculation of the single and generalized double error integral. This work is both a useful tool and a convenient reference source for all future computers of the error integrals. Its subject matter is both stimulating and suggestive of approaches to much more general types of integrals where the integrands are functions of the exponential type.

In connection with **A**, some preliminary computational data are tabulated separately in **B**.

Most of this report, p. 2–36, gives the first 30 or so coefficients in the Taylor expansions of $Rr(u)$, $Ri(u)$, $Rr^2(u) + Ri^2(u)$, $\int_0^u Rr(x)dx$, and $\int_0^u dx\int_0^x Rr(y)dy$ about the points $u = \frac{1}{2}(\frac{1}{2})3$, (about the point 0, coefficients are given for certain smooth parts of these functions, which are defined in the text). Most coefficients are given to 20D or more. On p. 41–44 are some 30 Taylor coefficients of $e^{-w^2}\int_0^w e^{y^2}dy$ about $w = [2(1)5; 15D]$. On p. 37 the following functions are tabulated: $Rr(u)$, $Ri(u)$, $Rr^2(u) + Ri^2(u)$, for $u = [5(.05)6.15; 13D]$. On p. 40, there are values of $e^{-w^2}\int_0^w e^{y^2}dy$ for $w = .05(.05).6, 1(.5)4, 5, 6, 6.5$, almost all to 15D, and of $e^{w^2}\int_w^\infty e^{-y^2}dy$ for $w = 3, 3.8, 4.1, 4.7, 5, 5.5, 6$, to 13D or more. On p. 38–39, the approximating polynomials $Q(n, u)$ and $P(n, u)$, where $Rr(u) + iRi(u) \cong \dfrac{Q(n, u)}{P(n, u)}$, are listed for $n = 4(4)28$ as functions of u and π.

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352[L].—J. COSSAR & A. ERDÉLYI, *Dictionary of Laplace Transforms.* Admiralty Computing Service, Department of Scientific Research and Experiment, London. Part 3B, no. SRE/ACS 108, June 1946, leaves VII 30–VII 70 + leaf "Index of notations" + t.p. leaf. 20.2 × 33.1 cm. Also *Corrections and Additions to SRE/ACS* 53, 68, *and* 71, June 1946, 9 leaves. All leaves are mimeographed on one side only. These publications are available only to certain Government agencies and activities.

This is the concluding part of the *Dictionary* (started in 1944) the earlier parts of which were reviewed in *MTAC*, v. 1, p. 424–425 and v. 2, p. 76. Part 3A contained the beginning of Section VII, with a Table of inverse Laplace Transforms, that is, a classification, according to $\phi(p)$ for the Elementary Functions, Gamma and Related Functions, and Functions defined by Integrals. Part 3B, the concluding part of Section VII, is the classification $\phi(p)$ for Bessel Functions, Confluent Hypergeometric Functions, Legendre Functions, and Hypergeometric Functions, etc. Practically all the corrections and additions are for

Section VI, Table of Laplace Transforms. We are told that some of these entries were discovered by comparing parts of the published *Dictionary* with N. W. McLachlan & P. Humbert, *Formulaire pour le Calcul Symbolique*, Paris, 1941, and that among correspondents H.B. "made the greatest single contribution to this list."

R. C. A.

353[L].—Robert C. Herman & Charles F. Meyer, "The thermoluminescence and conductivity of phosphors," *Jn. Appl. Physics*, v. 17, Sept. 1946, p. 748. 19.8 × 26.6 cm.

The article includes a table of $-Ei(-x) = \int_x^\infty e^{-u}du/u$, for $x = [15(.1)20; 6S]$, employed in the calculation of glow intensity curves, etc. It bridges a gap between tables of NBSMTP and BAASMTC,[1] for $0 < x \leqslant 15$, and of Akahira,[2] for $20 \leqslant x \leqslant 50$. The table was calculated from the asymptotic expansion

$$-Ei(-x) = e^{-x}[x^{-1} - x^{-2} + 2!x^{-3} - 3!x^{-4} + 4!x^{-5} - \cdots], \tag{1}$$

and second (central) differences are included under the heading Δ^2.

Recalculation by (1), of errors discovered by differencing, led to the following corrections:

At $x = 15.9$, *for* 7.38008×10^{-9}, *read* 7.38256×10^{-9};
$x = 19.1$, *for* 2.52729×10^{-10}, *read* 2.52793×10^{-10}.

There are corresponding errors in the differences in addition to the following:

At $x = 17.9$, *for* $.10031 \times 10^{-10}$, *read* $.09931 \times 10^{-10}$;
$x = 18.0$, *for* $.08843 \times 10^{-10}$, *read* $.08943 \times 10^{-10}$.

Modified second differences should have been provided.

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[1] NBSMTP, *Tables of Sine, Cosine and Exponential Integrals*, 2 v., 1940. BAASMTC, *Mathematical Tables*, v. 1, *Circular & Hyperbolic Functions, Exponential & Sine & Cosine Integrals . . .*, second ed., 1946.

[2] Takeo Akahira, *Tables of e^{-x}/x and $\int_x^\infty e^{-u}du/u$ from $x = 20$ to $x = 50$*, Institute of Phys. and Chem. Research, *Sci. Papers*, Tokyo, *Table no. 3*, 1929.

354[L].—Tokiharu Okaya, "Numerical tables of Tchebychef's q-functions and their integrated and derivated functions," Nippon Suugakubuturigakkwai *Kizi*, Tokyo, *Proc.*, s. 3, v. 23, 1941, p. 788–799. 18.1 × 25.6 cm.

The functions referred to in the title are

$$q_v(n, z) = v!2^{-v}\sum_{k=0}^{v}\binom{v+k}{k}\binom{v-n}{v-k}\binom{z}{k}$$

and were introduced by Chebyshev[1] to approximate a given function $f(z)$ defined for $0 \leqslant z \leqslant n-1$ by the linear combination

$$F_k(n, z) = a_0 + a_1q_1(n, z) + \cdots + a_{k-1}q_{k-1}(n, z). \tag{1}$$

This choice of the functions q minimizes the quantity

$$\sum_{r=0}^{n-1}[f(r) - F_k(n, r)]^2$$

and the coefficients $a_0, a_1, \cdots, a_{k-1}$ are determined by

$$S_v(n)a_v = \sum_{r=0}^{n-1} f(r)q_v(n, r),$$

where

$$S_v(n) = \sum_{r=0}^{n-1} \{q_v(n, r)\}^2.$$

The integral or derivative of $F_k(n, z)$ is an approximation to the integral or derivative of $f(x)$ and either can be obtained at once from (1).

The author refers to previous tables[2] of these functions, and notes that they contain misprints, but does not actually list them. The present tables (p. 790–798) give the exact (rational) values of I. $q_v(n, r)$ and of IV. its integral, of III. its derivative, and of II. $S_v(n)/u_v(n)$ for $n = 5(1)20$, $r = 0(1)19$ and $v = 1(1)6$; $u_v(n)$ is a certain numerical coefficient. T. I was prepared in collaboration with ZIRÔ YAMAUTI.

D. H. L.

[1] P. L. CHEBYSHEV, Akad. N., Leningrad, *Mémoires*, s. 7, v. 1, 1859, p. 1–24; and *Oeuvres*, St. Petersburg, v. 1, 1899, p. 473–498.

[2] (a) CHARLES JORDAN, London Math. Soc., *Proc.*, s. 2, v. 20, 1922, p. 322–325. The tables I–V are for $q_v(n, r)$, and table VI for $S_v(n)$, $v = 1(1)5$, $n = 2(1)20$, $r = 0(1)11$. (b) T. OKAYA [Methods of calculation, probability and statistics] (in Japanese), 1940. EDITORIAL NOTE: We have not been able to check this item whether it was an article in a periodical, or a book, by Okaya.

355[L].—JOSEPH B. ROSENBACH, EDWIN A. WHITMAN, & DAVID MOSKOVITZ, *Mathematical Tables*. Boston, Ginn, 1937. xl, 208 p., frontispiece, and duplicated p. 207–208 in pocket. 14.3 × 21 cm. Second edition, corrected and enlarged, 1943. xl, 212 p. $1.75.

T. XXVII, p. 183: $\Gamma(n)$, $n = [1.01(.01)2;\ 5D]$.
T. XXVIII, p. 183: $\log \Gamma(n)$, $n = [1.01(.01)2;\ 5D]$.
T. XXIX, p. 184–185: $n^{-\frac{1}{2}}$, $[n(n-1)]^{-\frac{1}{2}}$ each to 8D; $.67449(n-1)^{-\frac{1}{2}}$, $.67449[n(n-1)]^{-\frac{1}{2}}$, $.84535[n(n-1)]^{-\frac{1}{2}}$, $.84535[n(n-1)^{\frac{1}{2}}]^{-1}$ each to 5D; and all tables for $n = 2(1)100$.
T. XXX, p. 186: $(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}$, for $x = [0(.01)5;\ 5D]$.
T. XXXI, p. 187: $(2\pi)^{-\frac{1}{2}}\int_0^x e^{-\frac{1}{2}t^2}dt$, for $x = [0(.01)5;\ 5D]$.
T. XXXII, p. 188: $n!$ to five figures, and $\log n!$ to 5D, for $n = 1(1)60$.
T. XXXIII, p. 188: B_n to five figures, and $\log B_n$ to 5D, for $n = 1(1)60$.
T. XXXIV, p. 189: K and E for $\theta = [0(1°)70°(30')80°(12')89°(6')89°30'(3')90°;\ 4D]$.
T. XXXV–XXXVI, p. 190–197: $F(k, \phi)$, and $E(k, \phi)$, for $\theta = 5°(5°)90°$, $\phi = [1°(1°)90°;\ 4D]$.
T. XXXVII, p. 198–199: $P_n(x)$, for $n = 1(1)7$, $x = [0(.01)1;\ 4D]$.
T. XXXVIII, p. 200–201: $P_n(\theta)$, for $n = 1(1)7$, $\theta = [0(1°)90°;\ 4D]$.
For T. XXXIX–XLI, see *N*65.
T. XLII, p. 204: $\log x$ and $\log (1 + 0.1^n x)$, for $n = 1(1)11$, and $x = [1(1)9;\ 20D]$.
T. XLIII, p. 204: kM and k/M, for $k = [1(1)9;\ 24D]$.
T. XLIV, p. 205: $\log p$, p prime and <1000, to 20D.
T. XLVI, p. 207–210 (in new edition): Table of haversines. P. 211–212 (old edition p. 207–208): Proportional parts (also in pocket of new edition).

The changes in the first edition of the *Tables* were as follows:

P. 191, $F(75°, 37°)$, *for* 0.6913, *read* 0.6919.
P. 193, $F(60°, 60°)$, *for* 1.2125, *read* 1.2126.
P. 200, 53 corrections of $P_6(\theta)$ and $P_7(\theta)$.
P. 201, 2 corrections of $P_2(\theta)$, 2 of $P_4(\theta)$, 1 of $P_5(\theta)$, 16 of $P_6(\theta)$, 18 of $P_7(\theta)$. Pages 200–201 now agree with the corresponding tables in JAHNKE & EMDE (1945).
P. 203, $J_9(3)$, and fourth, fifth and eighth zeros of $J_1(x)$ are corrected.
P. 206, the digit in the thirty-second decimal-place approximation for π was correctly 0 in the first edition and incorrectly changed to 6 in the second edition.

R. C. A.

356[L, S].—F. B. PIDDUCK, *Currents in Aerials and High-Frequency Networks*. Oxford, Clarendon Press, 1946. iv, 97 p. 14 × 22.1 cm. 8s. 6d.

The tables include the following:

P. 70, $(1, s)$ for $s = [0(1)23; 5D]$, where $(n, s) = \int_0^\pi e^{isy} y^n dy$;
$(1, 23) = -35061200000 + 4410840000i$.
P. 86–91, Ci x, Si x for $x = [0(.01)10(.1)29.9; 5D]$.
P. 70, 96–97; in the treatment of aerials parallel to the earth the function $E_n^s(x) = C_n^s(x) + iS_n^s(x) = \int_0^\pi e^{isy} J_0(xy) y^n dy$ was introduced. For large values of x the function is expressible in terms of Lommel functions. There are tables of C_1^0, S_1^0, $C_1^{0\prime}$, $S_1^{0\prime}$, $C_1^1(x)$, $S_1^1(x)$, $\pi C_1^0(x) + S_1^{0\prime}(x) - C_1^1(x)$, $(x^2 - 1)^{\frac{1}{2}}$, for $x = [0(.05)2(.2)10; 4D]$.
P. 92–95, $\sin x$ and $\cos x$, for $x = [0(.01)7.99; 5D]$.

357[M].—MURLAN S. CORRINGTON, "Table of the integral $2\pi^{-1}\int_0^x \tanh^{-1} t\, dt/t$," *R. C. A. Review*, v. 7, Sept. 1946, p. 432–437. 14.9 × 22.7 cm. Compare UMT **51**, p. 184.

The integral $B(x) = \pi^{-1}\int_0^x \ln \left|\frac{1+t}{1-t}\right| dt/t = 2\pi^{-1}\int_0^x \tanh^{-1} t\, dt/t$ is involved in the computation of either the real or imaginary component of a minimum-phase-shift network, having the component given.[1] $B(x)$ is tabulated for $x = [0(.01).97(.005).99(.002)1; 5D]$ rounded off from computations to 8D [see *MTAC*, v. 2, p. 184]. The methods of computation and checking are explained in detail. After the difference test indicated that the computed values were all within one unit in the eighth decimal place, the values were rounded off to 5D. It is therefore hoped that the table is free from error.

Extracts from introductory text

[1] HENDRIK W. BODE, *Network Analysis and Feedback Amplifier Design*, New York, 1945, chapters 14–15.

358[M].—HARVARD UNIVERSITY, Computation Laboratory Reports for the U. S. Bureau of Ships: No. 2, June, 1944, *Evaluation of the Function* $S(b, h) = \int_0^h \sin (x^2 + b^2)^{\frac{1}{2}}\, dx/(x^2 + b^2)^{\frac{1}{2}}$; by H. H. AIKEN & R. V. D. CAMPBELL. 5 leaves mimeographed on one side of each. No. 10, Oct. 1944, *Evaluation of the Function* $C(b, h) = \int_0^h \cos (x^2 + b^2)^{\frac{1}{2}}\, dx/(x^2 + b^2)^{\frac{1}{2}}$, by H. H. AIKEN, H. A. ARNOLD, R. V. D. CAMPBELL, & R. R. SEEBER. 9 leaves mimeographed on one side of each. 20.3 × 26.7 cm. These tables are available only to certain Government agencies and activities.

These integrals arise in considerations connected with coupled antennae. Their values are tabulated for $h = .5(.5)6.5$, $b = [0(.1)6.3; 5D]$. In the table of $S(b, h)$ all arithmetic work was carried to 15D. Hence all tabular entries should be accurate. In the table of $C(b, h)$ an error of less than 3×10^{-6} is assured for all results, and in most cases it is less than 10^{-6}.

MATHEMATICAL TABLES—ERRATA

References have been made to Errata in RMT **338** (Nelshuler), **340** (Zimmermann), **341** (Elznic & Valouch), **342** (Marchin), **343** (Février), **344** (Elznic), **345** (Cherwell), **346** (Delfeld), **348** (Cunningham), **352** (Cossar & Erdélyi), **353** (Herman & Meyer), **354** (Jordan, Okaya), **355** (Rosenbach, Whitman & Moskovitz), N **63** (Leibniz), **64** (Womersley), **65** (Rosenbach, Whitman & Moskovitz), **66** (Crommelin, Meares).

In RMT 311, v. 2 p. 128, we listed five errors which E. L. KAPLAN had noted in Tables K and E in K. HAYASHI *Tafeln der Besselschen, Theta-, Kugel- und anderer Funktionen*,[1] Berlin, 1930, p. 72–81. In *Math. Reviews*, v. 7, 1946, p. 485, L. J. C. pointed out that Hayashi himself had corrected four of these errors [in a "Berichtigungen"-sheet for the volume, dated Fukuoka, 1932] and that the fifth error had been noted by us in *Scripta Math.* v. 3, 1936, p. 365. L. J. C. also adds that Kaplan listed six errors in E, all of which (and no more) were found by him in 1933, by differencing. Regarding Kaplan's remark that the last figure in MILNE-THOMSON's ten-figure table of the complete elliptic integrals was "unreliable" in the last figure, L. J. C. noted that "records show about 70 errors (in 200 values) of a unit each, one of two units, and none greater; this rather loose statement about a perfectly useful table is somewhat misleading."

[1] Compare review by L. J. C., *Math. Gazette*, v. 17, 1933, p. 283–284.

94. SCHUYLER M. CHRISTIAN, "Integration of $\sin^2 xdx/x$," *Phys. Rev.*, s. 2, v. 69, May 1 and 15, 1946, p. 546.

The table of 12 entries here given is reproduced below. This integral can be expressed as

$$\int_0^x \sin^2 xdx/x = \tfrac{1}{2}\gamma + \tfrac{1}{2}\ln 2x - \tfrac{1}{2}\operatorname{Ci} 2x,$$

where $\gamma = .5772\ 1566\ 49\ldots$ is Euler's constant, and $\operatorname{Ci} x = \int_\infty^x \cos t dt/t$.

Using the tables NBSMTP, *Tables of Sine, Cosine and Exponential Integrals*, v. 2, 1940, and NBSMTP, *Tables of Natural Logarithms*, v. 1, 1941, I arrived at the following results:

x	Christian	Corrington
.5	0.119 906	0.1199 0587
1	0.423 691	0.4236 9101
2	1.052 246	1.0522 4586
3	1.218 513	1.2185 1619
4	1.267 080	1.2671 1166
5	1.462 628	1.4626 2860
6	1.555 950	1.5559 5116
7	1.573 438	1.5734 3832
8	1.683 106	1.6820 0229
9	1.748 135	1.7555 3126
10	1.764 305	1.7642 6406
11	1.833 071	1.8333 0871

Thus eight of the twelve values given by Professor Christian appear to be incorrect.

MURLAN S. CORRINGTON

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EDITORIAL NOTE: Professor Christian is a professor of physics at Agnes Scott College, Decatur, Georgia.

95. FMR, *Index*. See *MTAC*, v. 2, p. 13–18, 136, 178–181.

A. P. 87, **5.16** log π to 101D, by PARKHURST 1889, is omitted; l. 8, *for* $\sqrt{6/\pi}$, *read* $\sqrt[3]{6/\pi}$. Perhaps it may be added that I have the following more extended fractional powers of π than three ms. values recorded in the *Index*: (a) p. 86, $\pi^{-\frac{1}{2}}$(60D+), 310D; (b) p. 87, $\pi^{\frac{1}{2}}$(51D), 110D; (c) p. 87, $\pi^{\frac{1}{3}}$(42D), 110D.

P. 196, l. −4, the upper limit in each integral should be infinity, not x.

P. 255, l. 1, *for* J_{-n-}, *read* $J_{-n-\frac{1}{2}}$.

JOHN W. WRENCH, JR.

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B. P. 250, move "4 dec." to the left.

P. 253, l. 5, *for* $x = 15(1)25$, *read* $16(1)25$; also $(16 - p)$ dec. is not correct for $p = 12(1)14$.

P. 389, under A. DeMorgan, 1837 and 1845 editions of "Theory of Probabilities" are noted, but there was still another edition in an undated volume of 1847, entitled *The Encyclopædia of Pure Mathematics: Comprising* [here follow titles of articles and names of authors of 12 contributions]. London and Glasgow, Richard Griffin & Co. Then come p. [303]–844, 1–208, 305–544+16 plates, reprinted from *Encyclopædia Metropolitana*.

A review of the *Index* in *Engineering*, v. 162, 18 Oct. 1946, p. 365, has the following: "The only misprints noted occur on p. 59, where the 039, corrected to 093, is itself correct and where the two zeros on the following line appear to be superfluous. The statement has been made, too—though we cannot recall its source—that the *Essay on Probabilities*, ascribed to De Morgan, on p. 389, and which certainly bears his name upon the title page, was not, in fact, written by him." The first error refers to the fifth and sixth decimals in the value of s_{10} in the table of Oettinger 1856; the numbers quoted under *For* and *Read* respectively, should simply be interchanged. The *Memoir of Augustus De Morgan* by his wife, with entries in the list of his Writings of not only the *Essay*, 1838, in the Cabinet Cyclopaedia series, but also of the long article on "Theory of Probabilities" in *Encyclopædia Metropolitana*, 1837, seems to render the reviewer's surmise untenable. Moreover, anyone familiar with De Morgan's characteristic writings in the field with which the volume deals would naturally assign the *Essay* to him, even if it were without title-page.

R. C. A.

96. M. B. Kraïtchik, "Les grands nombres premiers," *Mathematica* (Cluj), v. 7, 1933, p. 92–94. See RMT **36**.

The table contains what purports to be 94 primes successively numbered from no.1, 1 030 330 938 209, to no.94, the largest known prime $2^{127}-1$, containing 39 digits. There are the following errors in this table:

9 *For* $2^{80}-1$, *read* $2^{89}-1$

23 *Delete* this line, and *insert*[1] between nos. 6 and 7 the entry: $2748779069441 = 5 \cdot 2^{39} + 1$, factor of $2^{252} + 1$ (Seelhoff)

26 *Delete* this line. Entry = $550801 \cdot 23650061$

34 *For* $2^{185} + 1$, *read* $3^{185} + 1$

38 *For*[2] 76076..., *read* 76096...

42 *For* $5^{75} + 1$, *read* $5^{75} - 1$

46 *For* $2^{85} + 1$, *read* $2^{85} - 1$

51 *For* $10^{37} + 1$, *read* $10^{37} - 1$

52 *For*[2] 549767..., *read* 549797...

65 *Delete* this line. Entry = $1210483 \cdot 25829691707$

66 *For* $2^{111} + 1$, *read* $3^{111} + 1$

71 *For*[2] $16024\cdots$, *read* $16624\cdots$; *for* $2^{170} + 1$, *read* $2^{270} + 1$

88 *Delete* this line. Entry = $394783681 \cdot 46908728641$

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[1] An erratum noted by Th. Gosset, *Sphinx*, v. 3, 1933, p. 125.

[2] These typographical errata were given by N. G. W. H. Beeger, *Mathematica* (Cluj), v. 8, 1934, p. 212.

Editorial Note: In *Sphinx*, v. 3, 1933, p. 100–101, Kraïtchik gives what purports to be a list of 161 prime numbers greater than 10^{12}. Five of the above errata apply also to this list, namely: nos. 23, 26, 42, 65, 71 (first part).

97. S. LUBKIN & J. J. STOKER, "Stability of columns and strings under periodically varying forces," *Quart. Appl. Math.*, v. 1, 1943, p. 232–235. See *MTAC*, v. 1, p. 415.

The authors list 48 errors in the tables of this article in *Quart. Appl. Math.*, v. 4, Oct. 1946, p. 309–310.

98. F. ZICKERMANN, "Ueber Arbeitsmessung bei Wechselstrom mit besonderer Berücksichtigung des Drehstromarbeitsdynamometers von Siemens & Halske," *Elektrot. Z.*, v. 12, 1891; on p. 511 is a table of $f(x, y) = \tan y \cdot \tan (x - y)$.

This table is for $f(x, y) \leqslant 1$, $y = 0(1°)45°$, $x - y = [0(5°)90°; 3D]$. By differencing the following errata were readily found: $y = 9°$, $x - y = 75°$, *for* 0,581, *read* 0,591; $y = 20°$, $x - y = 65°$, *for* 0,770, *read* 0,781.

S. A. J.

UNPUBLISHED MATHEMATICAL TABLES

For other unpublished tabular numbers see RMT **346**, **348**; MTE **95**.

52[K].—STUART R. BRINKLEY, JR. & RUTH F. BRINKLEY, *Table of the probability of hitting a circular target.* Ms. prepared by, and in the possession of the authors, R. D. 3, Coraopolis, Pennsylvania.

It is readily verified that the probability that the point of impact of a missile aimed at the origin of a rectangular system of coordinates will lie within a circle of radius r whose center is at a distance R from the origin is

$$p(r, R) = e^{-R^2} \int_0^{r^2} e^{-t} I_0(2Rt^{\frac{1}{2}})\,dt,$$

if the probability distributions for the pair of rectangular coordinates are Gaussian; I_0 being the modified Bessel function of the first kind, zeroth order. A table has been prepared of the function $p(r, R)$, for $r = 0(.1)5$; $R = [0(.1)5; 5S]$. The construction of the table was made possible by a grant from the George Sheffield Fund of Yale University.

We undertook the construction of the table because of applications of the related functions

$$P(x, y) = e^{x+y} p(\sqrt{x}, \sqrt{y}),$$

which is the solution of

$$\frac{\partial^2 P}{\partial x \partial y} = P, \qquad P(o, y) = 0, \qquad P(x, o) = e^x - 1.$$

It appears to have numerous and rather diverse applications. It occurs, for example, in the theory of ion-exchange water softening columns (H. C. THOMAS, Amer. Chem. Soc., *Jn.*, v. 66, 1944, p. 1664f), the theory of heat exchange between a fluid and a porous solid (A. ANZELIUS, *Z. angew. Math. Mech.*, v. 6, 1926, p. 291f, and T. E. W. SCHUMANN, Franklin Inst., *Jn.*, v. 208, 1929, p. 405f), and in the extension of the latter theory to the case where the solid is generating heat, as in a catalytic chemical reaction (S. R. BRINKLEY, unpublished paper, in the press). However, the function P is considerably less well suited to tabulation than is the function p.

S. R. BRINKLEY, JR. & R. F. BRINKLEY

MECHANICAL AIDS TO COMPUTATION

In *Nature*, v. 158, 12 Oct. 1946, p. 500–506, is an interesting illustrated article by Professor D. R. Hartree, of the University of Cambridge, entitled "The ENIAC an electronic computing machine." (Dr. Goldstine's name is repeatedly misspelled; see *MTAC*, v. 2, p. 97–110.)

JACK LADERMAN & MILTON ABRAMOWITZ are the authors of an article "Application of machines to differencing of tables," in Amer. Statistical Assoc., *Jn.*, v. 41, June 1946, p. 233–237.

26[Z].—CLIFFORD E. BERRY, DOYLE E. WILCOX, SIBYL M. ROCK & H. W. WASHBURN, "A computer for solving linear simultaneous equations," *J. Appl. Physics*, v. 17, Apr. 1946, p. 262–272. 19.8 × 26.8 cm.

This paper is a description of the recent commercial model computer, brought out by the Consolidated Engineering Corporation of Pasadena, California, for solving twelve simultaneous linear equations. The mathematical theory of the machine based on the well-known Gauss-Seidel method of iteration is described briefly together with questions of convergence. References are given to recent articles in which necessary and sufficient conditions for convergence of the process are given. These are too complicated to be of much practical use, so the authors suggest two empirical rules for arranging the given equations for a satisfactory application of the method.

(1) Arrange the matrix of coefficients $A = (a_{ij})$ so that diagonal elements are large and other elements are small.

(2) If (1) is not feasible try to arrange A so that for every $i \neq j$, $|a_{ii}a_{ji}| < |a_{ii}a_{jj}|$. When (1) and (2) fail, another method, to be described in detail elsewhere, is suggested in which systems of equations in 2, 3, 4, $\cdots$, n unknowns are solved in turn.

The electrical analogue of a set of linear equations is, of course, a composite of circuits for adding and multiplying electrical quantities. In the present machine AC voltages, proportional to the sum and product of given voltages, are produced by simple circuits involving variable resistances. In operating the computer these circuits are balanced by observing a small cathode ray null indicator. The potentiometers corresponding to the coefficient of each one of the 12 equations are laid out along one element of a cylindrical switching turret. By rotating this cylinder by hand through successive multiples of 30°, about its horizontal axis, each one of the 12 sets of 13 potentiometers is made available, in turn, for setting at a horizontal window. Once these 156 potentiometers are set, the machine has been given the problem. According to the authors' examples this operation requires about 39 minutes for a 12 × 12 problem. Of course, if a series of such problems with the same matrix A is to be done (as for instance in the inversion of a matrix) it will be necessary to reset only the 12 constant terms each time. This requires only 3 minutes.

The next operation consists in adjusting the potentiometers corresponding to the unknowns $x_1, x_2, \cdots, x_{12}$. With the cylinder in its first position, the x_1 potentiometer is adjusted until the null indicator shows that a balance has been obtained. This means that the first equation is satisfied for this value of x_1 and the (arbitrary) other values $x_2, \cdots, x_{12}$. The cylinder is now rotated to position 2 and the x_2 potentiometer adjusted for balance. This process continues for perhaps several revolutions of the cylinder (depending upon the rapidity of the convergence of the process) until no further adjustments are required. The values of the x's at this time constitute the computer's answer to the problem.

Four examples are given, each employing 4 decimal places and showing the time required and the accuracy obtained in each case. The computer deals with numbers between + 1 and − 1 and can be set to within .1 percent. Such accuracy in the x's, of course, cannot be guaranteed since this will depend on what the authors rather vaguely call the "leverage" of the system, a quantity practically determined by the size of the determinant of A. In the rather favorable looking example 1 the maximum error is .0004. The computer appears

to be of an order of magnitude faster than desk calculators for this type of work. The time required to work Example 1 is given as 44 minutes of which only 2 were spent in adjusting the x's. This compares favorably with the 5-hour estimate given for desk machine work.

In problems where more accuracy is required, as in certain statistical work for example, it is possible to use first the computer and then to improve the solution thus obtained by using a desk calculator. In this way the electrical computer should be quite useful in any one of the host of problems which can be reduced to systems of linear equations.

D. H. L.

27[Z].—G. S. MERRILL, "Slide-disk calculator," *General Electric Rev.*, v. 49, no. 6, June 1946, p. 30–33. 22.3 × 29.5 cm.

This note is a well illustrated description of an ingenious combination of a sliding rule and a rotating disk for the computation of the standard deviation

$$\sigma = [x_1^2 + x_2^2 + \cdots + x_n^2]^{\frac{1}{2}} n^{-\frac{1}{2}}$$

of the set of deviates $x_1, x_2, \cdots, x_n$ from the arithmetic mean of n values. The quantities

$$x_1, (x_1^2 + x_2^2)^{\frac{1}{2}}, (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}, \cdots$$

are built up in succession geometrically by successive applications of the Pythagorean theorem. An alignment chart is then used to multiply the final square root by $n^{-\frac{1}{2}}$. The sliding rule permits one to set the device with the origin at the arithmetic mean of the values concerned so that one need not really calculate the deviates themselves. No statement is made as to the accuracy obtainable. This depends somewhat on the order in which the n values are introduced. In general the accuracy should be comparable with that obtained from an ordinary slide rule. Even a much cruder device would be useful for class room demonstration purposes.

D. H. L.

NOTES

63. EARLY TABLES OF FACTORIALS.—While tables of $N!$, by various authors, 1926 to 1936, $N = 20$ to 30, are reported in FMR, *Index*, p. 34, the following three very much earlier appearances of such tables may be recorded:

(a) G. G. LEIBNIZ, *Dissertatio de Arte Combinatoria*. Leipzig, 1666, p. 57. Table of $N!$ for $N = 1(1)24$ (erratum $N = 15$, *for* 1307874368000, *read* 1307674368000). Also, with the same erratum, in *Leibnizens mathematische Schriften*, ed. by C. I. GERHARDT, part 2, v. 1, Halle, 1858, p. 61.

(b) JOHN WALLIS, *A Treatise of Algebra*. . . . London, 1685, Part IV, p. 116. Table, without error, up to $N = 24$.

(c) JAMES DODSON, *The Calculator: being, correct and necessary Tables for Computation. Adapted to Science, Business and Pleasure*. London, 1747. P. 44, Table XXIII, up to $N = 30$.

S. A. J.

64. ERRATUM:—May I call attention to a rather amusing type of error made by J. R. WOMERSLEY in his article "Scientific computing in Great Britain," *MTAC*, v. 2, p. 114? In illustrating the very great power of the h^2-process in deferred approach to the limit, he states that its application to the determination of π from the perimeters of the inscribed square and

hexagon leads to the value of 3.18 "which is astonishingly accurate when we consider the crudeness of the two original approximations, namely 2.83 and 3.00."

As a matter of fact, its application does not give 3.18 but 3.14, which is even more remarkable! For if we assume that the error is inversely proportional to the square of the number of sides (a) $\pi - 2.83 = \frac{1}{16}K$, (b) $\pi - 3.00 = \frac{1}{36}K$. Elimination of K leads to (b) $9\pi/4 - 3.00 \times 9/4 = K/16$. Thus with (a), $\pi - 2.83 = 9\pi/4 - 6.75$, or $5\pi/4 = 3.92$, $\pi = 3.136$ or 3.14.

HERBERT E. SALZER

NBSMTP

65. *Guide* (*MTAC*, No. 7), SUPPL. 5 (for Suppl. 1–4, see v. 1, p. 403, v. 2, p. 59, 92, 190f).—J. B. ROSENBACH, E. A. WHITMAN & D. MOSKOVITZ, *Mathematical Tables*, 1937 (see RMT **355**) contains the following three tables involving $J_n(x)$:

T. XXXIX (p. 202), $J_n(x)$, $n = 0, 1$, $x = [0(.1)15; 4D]$;
T. XL (p. 203), $J_n(x)$, $n = 2(1)10$, $x = [1(1)24; 4D]$;
T. XLI (p. 203), zeros of $J_n(x)$, $j_{n,s}$, for $n = 0(1)5$, $s = [1(1)9; 3D]$.

These tables were compared with the corresponding tables of E. JAHNKE & F. EMDE, *Tables of Functions*. New York, 1943. The entries of T. XXXIX agree completely with those of J. & E., p. 156–163. In T. XL there is agreement with J. & E., p. 171–177, except for two entries: R. W. & M. shows $J_9(3) = 0.0000$, while J. & E. shows $J_9(3) = 0.00008440$; R. W. & M. has the corrected value (-0.0318, see *MTAC*, v. 1, p. 109) of $J_9(21)$. Since T. XLI agrees completely with J. & E., p. 168, there are 27 errors in each of the tables (*MTAC*, v. 1, p. 160, 282, 396, 398). These 27 errors are corrected in the 1945 edition of J. & E.

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EDITORIAL NOTE: In the second edition of the R. W. & M. work, 24 of the 27 errors in T. XLI still persist.

66. HUGE NUMBERS.—Testing with Lucasian sequences (see *MTAC*, v. 1, p. 333, 404; v. 2, p. 94) is extremely monotonous because the answer may not be known for months and the calculations have to be absolutely perfect throughout. So, I became rejuvenated by R. C. A.'s stimulating article on "A Huge Number," *N* **54**. The quoted value 4^{4^4}, namely "13407813 followed by 147 other figures" is too large by five units at least in the last place. In the year 1900 while working on π I used $\tan^{-1}\frac{1}{5}$ and prepared a non-consecutive table of powers of 2 which I later found agreed perfectly with the table given in 1853 by SHANKS (see *MTAC*, v. 2, p. 144). Dr. JOHN WRENCH JR., has recently verified my value of 2^{671}. In fact, therefore, $4^{4^4} = 2^{512} = 13407\ 80792\ 99425\ldots$ to 155S.

Now for CHR. WEISS and $N = 9^{(9^9)}$. His approximation of $\log N$, to 74D, is practically perfect, for his last three figures "700" really come from 699.919. . . . A closer approximation, to 250D, is the following:

3696	93099.	63157	03587	43543	09509	54829	29683	40024	68555	47962
40532	88963	42699	97525	70034	06999	19193	67130	50838	43111	13397
79021	94259	06692	14914	74561	62571	46159	47916	86083	85948	43688
35255	18207	03897	27927	06237	04405	18967	44041	41213	73418	07380
80236	22090	62682	55872	72564	31009	18799	97885.			

I prefer, in general, the base e to the base 10, hence I wrote

$\ln N =$ 8512 49820. 19441 64306 91970 00392 68915 27219 20072 90691
44735 83107 43540 06734 37562 69257 39691 92203 38560 80379 50645
35319 56155 21919 86091 20430 32813 69381 92937 99118 55468 39229
06220 99672 39479 89224 30182 40042 11595 68346 79044 49991 39094
45918 12193 88257 46708 62725 85558 73685 88936 17534 93881 48799
09364 92047 61035 24668 01099 15304 50610 94898 31096 87169 73851
16996.

From this value of $\ln N$ to 320D the calculation of N was made according to formulae (3) and (4) of the Introduction to my 137-place table (see *MTAC*, v. 1, p. 20). My result is $N = (4.28124\ 77317\ 57470\ 48036\ 98711\ 59305\ 63521\ 33905\ 54822\ 41443\ 51417\ 47537\ 23053\ 52388\ 74717\ 35048\ 35319\ 36652\ 99432\ 03337\ 50604\ 17533\ 64763\ 10007\ 80326\ 13904\ 69104\ 84) \times 10^{3869693099}$. The digit in the 137th decimal place, as obtained from my original work-sheets, is actually 3 followed by 9759, hence I have raised it to 4. The first 59 digits of N calculated earlier by Weiss check exactly with the above result.

I have looked up the article by J. W. MEARES in Br. Astr. Assoc., *Jn.*, v. 31, 1921, p. 277–278, relative to $9!^{(9!^{9!})}$. Towards everything in this note I take emphatic exception. 9! is no more one digit than $\Gamma(\frac{1}{2})$ is two digits. Worst of all, his boundaries for $9!^{(9!^{9!})}$ are both wrong. I found $\log 9!^{(9!^{9!})} = (3.58448\ 37219\ 01355\ 569\ldots) \times 10^{2017527}$. Since $10 > (3.58448\ldots) > 1$, $\therefore 10^{(10^{2017528})} > 9!^{(9!^{9!})} > 10^{(10^{2017527})}$. Again, since $10^{3.58448\ldots} = 3841.35\ldots$ roughly, $9!^{(9!^{9!})} = (3841.35\ldots)^{(10^{2017527})}$.

HORACE S. UHLER

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29 August 1946.

67. REVERSIBLE PRIME-PAIRS.—A reversible prime-pair is a set of two primes such that one is obtained by reversing the digits of the other, e.g. (3583, 3853). A similar reversible pair is of the type (929, 929), in which both numbers are identical, otherwise it is a dissimilar pair. In a table of such pairs given by GOPAL LAL MATHUR, in "Reversible prime-pairs," *Mathematics Student*, v. 13, 1945, p. 48, are given 65 prime-pairs below 5000, from 11 to 3853, 16 similar pairs and 49 dissimilar. This table is reproduced, in effect, in *Scripta Math.*, v. 11, 1945, p. 274.

QUERIES

20. SANG TABLES.—In what American Libraries may the following works of EDWARD SANG (see *MTAC*, v. 1, p. 368–370) be consulted?

(a) *A New Table of Seven-Place Logarithms of all Numbers from 20 000 to 200 000*, second issues improved. Edinburgh 1878, and London 1883. There are copies of the latter in the British Museum and the Bibliothèque Nationale. Where also may the 1915 reprint be found?

(b) *Life Assurance and Annuity Tables*, v. 2, "for every combination of two lives." London, 1859. In the British Museum there is a copy of both volumes of this work, so highly commended by A. DE MORGAN.

R. C. A.

QUERIES—REPLIES

27. TABLES TO MANY PLACES OF DECIMALS (Q1, v. 1, p. 30; QR1, 2, 3, 4, v. 1, p. 30–31, 67–68, 99–100; also p. 309–312).—In discussions of the work of the National Bureau of Standards Mathematical Tables Project (NBSMTP)[1], the question of the number of decimal places to which tables should be computed has frequently arisen. Certain persons interested in the Project have expressed the opinion that the number of decimal places should almost always be severely limited because of considerations relating to the applications, and have expressed the view-point that the Project has been open to criticism in the past because some of its tables have been computed to a large number of decimal places. The purpose of this note is to explain the present attitude of the Project toward the matter in question.

In the first place, it must be admitted that there is plenty of precedent for computing certain types of tables to many decimal places. To cite a few examples, the values of logarithms are given in the *Logarithmetica Britannica* to 20D; the values of the natural trigonometric functions are given by ANDOYER to 15D; the entries in most of the tables by VAN ORSTRAND are given to at least 23D, and in some cases to as many as 62D; the values in the tables of exponentials by F. W. NEWMAN are given to 12D; the probability functions are given by BURGESS to 15D; a short table of $\sin x$ and $\cos x$ was given by J. PETERS to 21D; the tables of the polygamma functions have been given by H. T. DAVIS to 12D, or more; the table of $\Gamma(x)$, $\log \Gamma(x)$ and $\Psi(x) = \Gamma'(x)/\Gamma(x)$ are given by H. T. DAVIS to 12D, or more; GLAISHER tabulated values of the sine, cosine and exponential integrals to 18D; HAYASHI gives 12-place values of Bessel Functions of fractional order, 12-place values of $J_0(x)$ and $J_1(x)$, and 12-place values of elliptic integrals; The *Jn. Appl. Physics*, Feb. 1946, contains a short table by the late Professor HARRY BATEMAN to 15D.

Thus on the basis of such circumstantial evidence, there seems to have been a genuine need, at least in the past, for tables computed to a large number of D. Does this need still exist? To answer this question, consider first the following typical situations encountered by professional computers in day-to-day operations:

(a) *Use of recurrence relations:* It is frequently necessary to employ recurrence formulae several times in succession. In such cases it is vital to begin with an adequate number of D in order to offset the loss of S inherent in the process.

(b) *Computation of derivatives:* It is frequently necessary to resort to numerical evaluation of the derivatives of analytic functions, since the computation from the explicit expression of these derivatives would involve a prohibitive amount of labor. In such cases the accuracy of the derivatives (particularly those of high order) would be inadequate unless the functional values used in the evaluation of these derivatives are known to a sufficiently large number of D. To illustrate the need for numerical differentiation, suffice it to mention the evaluation of the derivatives of the spherical scattering functions in the Mie theory for purposes of evaluating these functions for complex indices of refraction. (See *MTAC*, v. 2, p. 140–143).

(c) *Tabulations from leading differences:* When the value of a function is built up from its leading differences, the accuracy of the computed values

increases with the accuracy of the initial entries which give rise to the leading differences in question.

(d) *Sequences of arithmetic operations:* There is an inevitable loss of significant figures in any arithmetic process involving several operations. In such cases, the error arising from the accumulation of rounding-off errors can be considerable unless the data are given to a sufficiently large number of D. Very often loss of S is inherent in certain formulae even when the number of operations involved is relatively small. It is true that frequently the loss of S inherent in a particular method of computation may be obviated by choosing other methods. However such methods may involve more labor than the original method based upon the knowledge of certain functions to an adequate number of D.

Consider now the needs of the users of mathematical tables who are not professional computers. The requirements of this group are the determining factor.

Many of the situations, of the type outlined above which occur in connection with professional tabulation work, are frequently encountered by applied mathematicians in the everyday use of tables. Significant figures are lost in mathematical computations by the use of recurrence formulae to compute non-tabulated functions from tabulated ones. They are lost by chains of arithmetical processes. A particularly serious loss occurs when high powers of numbers are obtained by logarithms (unless the logarithms are known to a sufficient number of places), or when two nearly equal terms are subtracted from each other. In the evaluation of derivatives and turning points of functions, there is an inevitable loss in S. [Further discussions along these lines by HARRY BATEMAN, C. R. COSENS, and J. C. P. MILLER will be found at references given at the beginning of these notes.]

Thus it would seem that there is still a definite need for mathematical tables computed to a large number of decimal places for at least some of the basic functions.

The Mathematical Tables Project does not specialize in tables to a large number of D. Of all the tables prepared by the Project only the exponentials, the natural logarithms and the probability functions are computed to very large number of D (18, 16, and 15 respectively). The tables of $\tan^{-1} x$ and of $\sin^{-1} x$ are given to 12D.

These tables constitute only a small fraction of the output of the Project, both in published and unpublished tables, as well as in tables of a specialized nature prepared for the war effort. In the last mentioned tables, the number of D or S is always less than 10 and sometimes as small as 2. It is interesting to note, however, that the first editions of all the basic tables computed to a large number of D are almost entirely sold out.

It seems entirely within the realm of possibility that the new electronic digital computing machines now being developed will render certain types of mathematical tabulation obsolete. It is certainly expected that these machines will render it possible to use a direct numerical approach to the mathematical formulations of many physical problems, rather than the classical approach involving reduction of the mathematical formulation to previously tabulated functions. But it will be some years before these machines become widely available, and even when they do, it would appear at present that function tables may play an important role in their most

efficient use. If that is true, then of course the machines will lose significant figures in just the same way that the hand computers do. In particular, it is considered probable at present that recurrence relations will frequently be used in programming, so if function tables are to be used in connection with such machines, many S in these tables may be indispensable.

In the planning of the work on basic mathematical tables, the NBSMTP has had the advantage of a continual exchange of views with outstanding mathematicians, physicists and engineers both here and abroad. The final decisions regarding the scope of the tables in question, the range and interval of the arguments and the accuracy of the entries (number of D or S) reflect the judgment of these authorities.

A. N. LOWAN

[1] EDITORIAL NOTE: The Mathematical Tables Project has been a unit of the National Bureau of Standards since 1941. Beginning with 1947, along with other *MTAC* changes in notation, we shall refer to this Project by the symbol NBSMTP (like BAASMTC) and no longer use NYMTP.

CORRIGENDA

V. 1

P. 228, $\mathbf{A}_6$, heading, *read* $J_n(x)I_{n+1}(x) + J_{n+1}(x)I_n(x) = J_n(x)I_{n-1}(x) - J_{n-1}(x)I_n(x) = J_n(x)I_n'(x) - J_n'(x)I_n(x)$.

P. 233, **B** 1, *for* 14, *read* 1.4.

P. 235, **E** 1, *for* $1\frac{1}{2}(\frac{1}{2})10\frac{1}{2}$, *read* $1\frac{1}{2}(1)10\frac{1}{2}$.

P. 238, $\mathbf{C}_1$ 7, *for* $-4(.4)+4$, *read* $-4(.2)+4$.

P. 246, l. 2, *for* $G(x)$, *read* $G(x)/\pi$.

P. 252, l. 17, *for* i^n, *read* i^{-n}.

P. 254, **A** 3, *for* 10D, *read* 4D.

P. 255, $\mathbf{B}_4$ heading, *for* $-\frac{1}{2}\pi$ her' x, *read* $\frac{1}{2}\pi$ her' x; $\mathbf{B}_4$ 11, *for* $-$her' x, hei' x, *read* her' x, $-$hei' x.

P. 257, **E**6, *for* $\theta_0(x)$, *read* $-\theta_0(x)$; **E**12, *for* $N_1(x)$, *read* $(2/\pi)N_1(x)$; **E** 13, 14, *add* values of $x = 0(.1)10$.

P. 261, $\mathbf{A}_1$ 6, *for* $(x/\pi)^{\frac{1}{2}}$, *read* $(2x/\pi)^{\frac{1}{2}}$.

V. 2

P. 60, l. 14, *for* 266, *read* 288.

P. 103, l. 2, *for* and, , *read* and

P. 106, l. 1, *for* open certain circuit lines, *read* open circuit certain lines.

P. 125, **306**(d), *for* T_n^2, *read* T_n^1.

P. 136, l. 22, *for* $C_1(u)$, *read* $c_1(u)$.

P. 139, l. 10, *for* TALLQUIST, *read* TALLQVIST.

P. 149, l. 15, *for* stopped-wheel, *read* stepped-wheel.

P. 163, l. 2, *for* This edition was soon, *read* This edition, soon; l. 2, 4, *for* p. xxx, *read* p. XXX.

P. 167, RMT **323**, formula (2), *for* $f(n)$, read $f(x_n)$.

P. 168, formula (4), *for* $f(x_n)$, *read* $f(n)$.

P. 175, **334**, l. 2, *for* Development, *read* Experiment.

P. 194, **15**, *for* **306** (G), *read* **306**(b); **25** *for* 266, *read* 288.

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CLASSIFICATION OF TABLES, AND SUBCOMMITTEES

A. Arithmetical Tables. Mathematical Constants
B. Powers
C. Logarithms
D. Circular Functions
E. Hyperbolic and Exponential Functions
Professor DAVIS, *chairman*, Professor ELDER, Professor KETCHUM, Doctor LOWAN

F. Theory of Numbers
Professor LEHMER

G. Higher Algebra
Professor LEHMER

H. Numerical Solution of Equations
J. Summation of Series

I. Finite Differences. Interpolation
K. Statistics
Professor WILKS, *chairman*, Professor COCHRAN, Professor EISENHART, Professor FELLER, Professor HOEL

L. Higher Mathematical Functions
M. Integrals

N. Interest and Investment
O. Actuarial Science
Mister ELSTON, *chairman*, Mister THOMPSON, Mister WILLIAMSON

P. Engineering

Q. Astronomy
Doctor ECKERT, *chairman*, Doctor GOLDBERG, Miss KRAMPE

R. Geodesy
S. Physics, Geophysics, Crystallography
T. Chemistry
U. Navigation

V. Aerodynamics, Hydrodynamics, Ballistics

Z. Calculating Machines and Mechanical Computation
Professor CALDWELL, *chairman*, Doctor COMRIE, *vice-chairman*
Professor AIKEN, Professor LEHMER, Doctor MILLER, Doctor STIBITZ, Professor TRAVIS, Mister WOMERSLEY

EDITORIAL AND OTHER NOTICES

The addresses of all contributors to each issue of *MTAC* are given in that issue, those of the Committee being on cover 2. The use of initials only indicates a member of the Executive Committee.

Due to the enlargement of *MTAC* and publication of illustrations, beginning with 1947 the subscription price for each *calendar year* is $4.00, *payable in advance;* ordinary single numbers $1.25. Earlier ordinary single numbers each $1.00, and all numbers for each of the years 1943 to 1946 inclusive, $3.00. Special single number 7, *Guide to Tables of Bessel Functions*, $1.75, and number 12, $1.50. All payments are to be made to National Academy of Sciences, 2101 Constitution Avenue, Washington, D. C. No reductions are made to Libraries or to Booksellers. No sample copies are distributed, but detailed descriptive circulars will be sent upon application.

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